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A differential equation model of North American cinematic box-office dynamics

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A new mathematical model is presented for the box-office dynamics of a motion picture released in North America. Though previous work on this problem has usually involved probabilistic methods, the new model for describing cumulative box-office gross uses a continuous-time, differential-equation approach. This model, which consists of a system of three nonlinear, coupled, ordinary differential equations, incorporates the effects of marketing and advertising expenses, audience reaction, critical reviews, and previous box-office behavior, among other factors. Analytical asymptotic results are presented for various parameter regimes. In the general case, the model must be solved numerically. Numerical simulations are tested against actual revenue data from several recent movies to analyse the model's accuracy. An algorithm for practical usage of the model is presented.

Keywords: box office; film gross; motion pictures; deterministic models; perturbation methods; numerical simulations.

1. Introduction

In this paper we present a mathematical model for the dynamics of the box-office gross of a film released in North America. According to the *1999 Economic Review* by the Motion Picture Association of America, there were 442 new feature films released in 1999 on 37 185 screens. The North American box-office gross for calendar year 1999 was 7·45 billion US dollars, with an average admission price of US$5·08 (MPAA, 1999). Complicating our task is the fact that even though there were only 442 films released, the gross for each film is really a sum of small contributions from a large number of units. These units consist of either tens of thousands of screens or tens of millions of individual filmgoers, and our goal is to mathematically describe and understand the time-dependent behavior of each.

With such a large amount of money at stake, clearly a predictive model for the factors underlying box-office gross would be of great value to the studios (*distributors* in the

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literature), who are responsible for the production and distribution of films. Such estimates could help a distributor plan advertising budgets, contractual stipulations, etc. However, the total gross for a movie is only a rough estimate of the revenue the distributor will receive, since the distributor receives money through time-varying contracts with theatre owners (*exhibitors* in the literature). After the US Supreme Court antitrust decision in *US v. Paramount Pictures* (1948), exhibitors and distributors must be corporate strangers. The interested reader may consult other sources (De Vany & Eckert, 1991; Litman, 1983; Smith & Smith, 1986) for a more thorough discussion of the impact of the Court's decision on the modern film industry.

The typical rental contract (*exhibition license* in the literature (De Vany & Walls, 1997)) typically includes the duration the exhibitor may show the film, an agreement for how box-office revenue will be shared between the exhibitor and the distributor, and an exclusivity provision. The initial period of the rental contract is usually for four weeks, during which the bulk of the revenue (usually 60–70%) goes to the distributor. After this period, the exhibitor may continue to show the film, but now the split is reversed, with the bulk of the revenue (usually 60–70%) going to the exhibitor. Thus, the actual amount a distributor will realize is intrinsically tied to the weekly gross of a film.

Complicating matters further are additional stipulations of a typical contract. For instance, there is often a standard clause which sets a threshold weekly box-office total (called 'the house nut') over which the distributor gets 90% of the gross. If a film 'cracks the house nut' (performs in excess of the threshold), the exhibitor may ask for permission to screen the film on additional screens. These additional provisions will necessarily complicate the calculation of a studio's take, and hence in this paper we ignore them, focusing instead on predicting weekly grosses for the film. Once a successful model for this has been established, it is relatively straightforward to predict a studio's take. This summary of the nature of the contractual relationship between exhibitors and distributors is adapted from a more detailed exposition in De Vany & Walls (1996).

Clearly the gross for a film depends on the number of screens on which it is shown and the average number of people per screen who attend each showing. Unfortunately, since the latter factor depends on individual preferences, the problem of describing the dynamics of the overall box-office gross is a difficult one, as previous authors have noted (Litman, 1983; Smith & Smith, 1986; De Vany & Walls, 1999). The difficulty itself has been commented on by such luminaries as acclaimed screenwriter William Goldman (*The Princess Bride*, *Butch Cassidy and the Sundance Kid*), who has famously said that in Hollywood, ' ... nobody knows anything' (Goldman, 1983).

Nonetheless, this difficult problem of quantitatively modeling box-office dynamics has engendered fruitful academic research. Smith & Smith (1986), writing in *Applied Economics*, have described the motion picture as 'one of the best examples of the differentiated product envisioned in conventional models of monopolistic competition.' Almost all of the research published prior to our paper brings probabilistic and statistical approaches to bear on the question of quantitatively describing motion picture revenue. One prolific author in this subject area is Economics Professor Arthur S. De Vany of the University of California at Irvine, who has exclusively used stochastic branches of mathematics in his approach to modeling cinematic box-office (De Vany & Walls, 1996, 1997, 1999).

The starting point for most of the models is to compare the release of a motion picture to the celebrated Bass model of the release of a new consumer durable product (Bass, 1969). The Bass model involves three unknown parameters: the number of initial purchases of the product, a coefficient of innovation and a coefficient of imitation. It is possible to reformulate the stochastic version of the Bass model as one involving an ordinary differential equation with an initial condition. The solution of this system, with given values of the parameters, produces the now-famous S-curve which represents the rate of adoption of a new product over time.

Another accomplished researcher in this area is Professor Jehoshua Eliashberg of the Wharton Business School at the University of Pennsylvania. Sawhney & Eliashberg (1996) use a two-parameter exponential distribution and a three-parameter generalized gamma distribution to model the process of movie selection by the consumer as two steps: the time to decide to see a new movie and the time to act on the viewing decision. More recently, Eliashberg *et al.* (2000) have developed a program called MOVIEMOD which can be used to predict box-office gross by using only pre-release data. The mathematical approach involves the probabilistic topic of Markov chains. De Vany & Walls (1996) use a Bayesian process of consumer demand for successful movies which they show leads to a Bose–Einstein distribution in box-office revenue. What all these models have in common is a desire to use probability distributions as the tool of choice for mathematical description of the situation at hand. In fact, in De Vany & Walls (1999) the authors model movies as stochastic dynamic processes, specifically a Lévy distribution. Their paper is a mathematical confirmation of the ‘terrifying’ result that most of Hollywood knows instinctively (and which is summarized in the epigrammatic Goldman quote): the probability distribution of box-office revenue has infinite variance.

For a number of reasons we have chosen to model the problem using differential equations. One obvious difficulty in the probabilistic-bound approaches is the immediate question of choosing the appropriate probability distribution for the particular processes being modeled. With the exception of Bass (1969), the other papers discussed use different probability distributions in each of their respective models of cinematic box-office. Our approach avoids this question altogether. Motivated by the success of the Bass model, we feel that applying ODE models to box-office dynamics provides a fresh perspective which can supplement the probabilistic models.

In this paper we formulate a model based on a deterministic approach where the revenue can be predicted based on empirical laws. The probabilistic nature of the problem is then folded into parameters in the deterministic model, parameters which vary from film to film. This idea of moving the source of the uncertainty from the model itself to the parameters in a deterministic model is certainly not new. It has been applied extensively to areas such as epidemics, where the probabilistic event is the spread of the disease when two people come in contact (Murray, 1993). The diffusion equation can be derived as the deterministic limit of a random walk process (Lin & Segel, 1988). The idea has even been used to give a deterministic interpretation of a coin toss (Keller, 1986).

In Section 2 we present our mathematical model of cinematic box-office dynamics. This involves deriving the governing differential equations and the introduction of relevant parameters. The result is a system of three coupled dimensionless nonlinear ordinary differential equations. We also discuss the various types of parameters used in our model and how they can be estimated.

In Section 3 we present analytical results. We demonstrate the difficulty of solving the system of differential equations analytically, and then present asymptotic results for the model in various parameter regimes. These results help to indicate the effects of varying certain parameters in the model. Where meaningful, we indicate the relationship between the asymptotic results and the corresponding business significance.

In Section 4 we present numerical results for the model and indicate how varying certain dimensionless parameters affects the computed solutions. We also discuss the actual estimation process used to determine the parameters.

In Section 5 we present our numerical results for movies in three distinct classes: a blockbuster film, a film that performed moderately well, and a film that performed poorly. We show that the model presented here is flexible enough to produce encouraging results for different classes of films with various box-office performance.

In Section 6 we present an algorithm for managers to self-implement the model. The algorithm can then be used by managers to make informed choices on how to maximize profits by using the predictive ability of the model of the cinematic box-office.

2. Developing the model

2.1 *Governing equations*

If the total gross of a movie at time $\tilde{t}$ is given by $\tilde{G}(\tilde{t})$, then the ultimate goal of our model is to obtain a value for $\tilde{G}(\infty)$, the final gross. The rate of change of the gross is the revenue function $\tilde{R}(\tilde{t})$:

$$\frac{\mathrm{d}\tilde{G}}{\mathrm{d}\tilde{t}} = \tilde{R}. \tag{1}$$

$\tilde{R}$ is written in units of dollars/time, and hence can be related to the weekly grosses widely circulated throughout the news media. However, (1) models the system as a continuous process, rather than a discrete one. Thus, for a true quantitative analysis one would have to compare averages of our continuous data to the discrete data released by the media each week.

The revenue $\tilde{R}(\tilde{t})$ will simply be the product of $\tilde{A}(\tilde{t})$, the amount of money earned per screen per week, and $\tilde{S}(\tilde{t})$, the number of screens on which the film is being presented. Thus,

$$\frac{\mathrm{d}\tilde{G}}{\mathrm{d}\tilde{t}} = \tilde{S}\tilde{A}. \tag{2}$$

At first glance, it may seem odd to treat $\tilde{S}$ and $\tilde{A}$, which is a ratio with denominator $\tilde{S}$, as separate variables in our model. Though clearly not independent in the statistical sense, this is not relevant in the differential equation context, where we consider only the parameters in the equations to be probabilistic. We can transform variables as we see fit, so we choose to use $\tilde{S}$ and $\tilde{A}$ since the raw data from the studios comes in this form. This transformation then requires that sometimes their product will appear together in the equations, as in the right-hand side of (2).

We know the initial gross $\tilde{G}(0)$ is zero, and clearly the final cumulative box-office gross

$\tilde{G}(\infty)$ can be computed by integrating the weekly revenue $\tilde{R}(\tilde{t})$ over all time, so we obtain

$$\tilde{G}(\infty) = \int_0^\infty \tilde{R}\,\mathrm{d}\tilde{t} = \int_0^\infty \tilde{S}\tilde{A}\,\mathrm{d}\tilde{t}. \tag{3}$$

$\tilde{A}$ and $\tilde{S}$ will be affected by audience reaction to the film. To model this effect, we introduce a measurable characteristic $H_{\%}$, the percentage of viewers who have a negative reaction to the film. This fraction is related to the 'favorability rating' studios compute after test screenings. It is interesting to note that distributors often will not release films with favorability ratings less than 80% (Puig, 2000), which in our model corresponds to an $H_{\%}$ of 0·2.

Note that $H_{\%}$ is essentially a psychological variable; that is, it is an inherent measure of how the average person will react to a film, and as such is a metric for the quality of the film as experienced by the general public. Since we are considering nationwide averages, we expect this proportion to be a constant. (The effect of bad word of mouth on a film, which is certainly not constant, is modeled in Section 2.2.4.) There are rare exceptions, such as *Star Wars* or *Lord of the Rings* movies, where a core set of fans, with predictably different reactions from the general public, may attend the first screenings, impervious to reviews or marketing.

We denote the total number of viewers who have a negative reaction to the film at time $\tilde{t}$ by $\tilde{H}(\tilde{t})$. Note that $\tilde{H}$ includes all viewers since release who have not liked the film. Our model assumes that $\tilde{H}$ is governed by the evolution equation

$$\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\tilde{t}} = \frac{H_{\%}\tilde{S}\tilde{A}}{P}, \qquad \tilde{H}(0) = 0, \tag{4}$$

where P is the average ticket price. (Note that $\tilde{A}/P$ is the number of people who attend a movie at a particular screen.) From (2) and (4) we obtain the expressions

$$\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\tilde{t}} = \frac{H_{\%}}{P}\frac{\mathrm{d}\tilde{G}}{\mathrm{d}\tilde{t}} \quad \text{and} \quad \tilde{G} = \frac{P}{H_{\%}}\tilde{H}. \tag{5}$$

At first glance it may seem somewhat counterintuitive that $\tilde{G}$ is proportional to $\tilde{H}$, the number of people who *did not* like the film. But both $\tilde{G}$ and $\tilde{H}$ are proportional to the total number of people who saw the film, and hence they are both proportional to one another.

Examination of the data indicates that, in the main, the two components of revenue, namely $\tilde{A}$ and $\tilde{S}$, decrease with time since release. We choose the simplest model for this decrease, specifically exponential decay. Then $\tilde{A}$ and $\tilde{S}$ will each have an inherent, natural decay rate, denoted by $\tilde{\alpha}_A$ and $\tilde{\alpha}_S$, respectively. Thus we expect the governing equations for $\tilde{A}$ and $\tilde{S}$ to resemble

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tilde{t}} = -\tilde{\alpha}_A\tilde{A}, \qquad \frac{\mathrm{d}\tilde{S}}{\mathrm{d}\tilde{t}} = -\tilde{\alpha}_S\tilde{S}, \tag{6}$$

under certain characteristic conditions.

Scilicet, we declare that the decay rate for $\tilde{A}$ will be $\tilde{\alpha}_A$ when

#1: no one who sees the movie goes to see it again;

#2: reviews of the film are mixed in such a way that they do not affect attendance;
#3: the film has no advertising or marketing budget;
#4: no one has a negative reaction to the film (100% favorability rating).

Clearly these are artificial conditions, and in the next section we will introduce additional terms into (6) to model what happens as each of these conditions is violated. We note that with the assumptions stated, $\tilde{\alpha}_A$ includes effects of the 'drawing power' of the cast, the film's genre, release date, and budget, as well as other factors which affect attendance but which we have not explicitly enumerated.

Similarly, the decay rate for $\tilde{S}$ will be $\tilde{\alpha}_S$ when no one is attending the film (i.e. $\tilde{A} = 0$), and hence we expect it to be quite high. In other words, if no one is attending the film, exhibitors will very rapidly cease showing it, and thus $\tilde{S} \to 0$. Again, $\tilde{\alpha}_S$ includes other effects not explicitly enumerated in the model. However, in contrast to $\tilde{\alpha}_A$, which includes effects of various factors on public perception and subsequent attendance, $\tilde{\alpha}_S$ includes effects of various factors on exhibitor action.

2.2 *Decay rate of attendance*

We now model each of the terms listed above that influence the decay rate of $\tilde{A}$. As noted above, $\tilde{A}$ is proportional to the weekly attendance at a single screen, and thus arguments made about attendance carry over to $\tilde{A}$. The true evolution equation for $\tilde{A}$ is assembled by considering exactly how each of the enumerated parameters influences the inherent decay rate $\tilde{\alpha}_A$.

2.2.1 *Relaxing restriction #1: repeat viewing.* Let D be the number of times the average filmgoer would see the film. Clearly D has a lower bound of 1, but for certain blockbuster movies can increase many percentage points above that.

We note that the solution to (6) is given by a simple exponential, and that the total revenue per screen is

$$\int_0^\infty \tilde{A}(\tilde{t})\,\mathrm{d}\tilde{t} = \int_0^\infty \tilde{A}(0)\,\mathrm{e}^{-\tilde{\alpha}_A \tilde{t}}\,\mathrm{d}\tilde{t} = \frac{\tilde{A}(0)}{\tilde{\alpha}_A}.$$

However, if each viewer sees the film D times, the total revenue per screen is given by

$$\int_0^\infty \tilde{A}(\tilde{t})\,\mathrm{d}\tilde{t} = D\frac{\tilde{A}(0)}{\tilde{\alpha}_A} = \int_0^\infty \tilde{A}(0)\,\mathrm{e}^{-\tilde{\alpha}_A \tilde{t}/D}\,\mathrm{d}\tilde{t}.$$

Thus, if the attendance *increases* by a factor k, then the decay rate *decreases* by the *same* factor k. It is this property of exponential behavior which we use to modify our model for the decay rate to include the effects of D.

2.2.2 *Relaxing restriction #2: the effect of reviews.* Similarly, let movie critics affect the attendance of the film in such a way that $1 + \epsilon$ times as many people go as if there were no reviews. (We expect ϵ to be a very small number, perhaps $|\epsilon| \leqslant 0{\cdot}05$.) Interestingly, Eliashberg & Shugan (1997) claim that while critical reviews are correlated with cumulative box-office receipts, they have almost no effect on early box-office performance. This makes sense, as it often takes a while for critical reviews to be disseminated to the bulk of the moviegoing public. If ϵ is positive, then this will cause a slight reduction in the intrinsic decay rate. By the argument above, we see that including factors #1 and #2 will then change the first equation in (6) to

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tilde{t}} = -\frac{\tilde{\alpha}_A}{D(1+\epsilon)}\tilde{A}.$$

Careful readers will note that we could have just defined $\tilde{\alpha}_A = \tilde{\alpha}_A/D(1+\epsilon)$ earlier and eliminated this discussion entirely. However, this would have been equivalent to lumping reviews and repeat viewings in the same category with genre, cast, budget, etc. But the nature of those parameters is quite different. From a large database of previously made films with similar genres, casts, budgets, release dates, etc., one should be able to obtain a reasonable *a priori* estimate for $\tilde{\alpha}_A$. However, the parameters D and ϵ can be estimated with a degree of accuracy only after a movie has been released. Therefore, these effects are listed separately. (For more discussion of parameter estimation issues, see Section 2.5.)

2.2.3 *Relaxing restriction #3: advertising.* A large portion of the average movie's budget is devoted to advertising (MPAA, 1999). Pre-release advertising is purchased to drive up the value of $\tilde{A}(0)$, while post-release advertising is purchased to keep the weekly attendance as high as possible. Thus the effect of increasing the rate of money M spent on advertising and marketing is to decrease the decay rate. This is based on our assumption that $\tilde{A}$ will always decrease. In practice it is possible for $\tilde{A}$ to increase.

The effectiveness of the marketing dollar is going to be diluted by the number of markets in which advertising is purchased. Since we are examining nationwide averages, the number of markets translates into the number of screens, so the relevant variable is really $M/\tilde{S}$.

Apart from the most recent effort by Eliashberg *et al.* (2000), most of the other attempts at modeling cinematic box-office have not included the effect of advertising explicitly in their mathematical formulation. In fact, even though the Bass model and most models derived from it (as summarized in Mahajan *et al.* (1990)) can be written in ordinary differential equation form, most investigators (Eliashberg *et al.*, 2000; De Vany & Walls, 1996, 1999; Neelamegham & Chintagunta, 1999) adopt a strictly Bayesian approach.

We assume that the effect of advertising should appear functionally as a factor multiplying the decay rate. As implied by #3, the factor should be unity when the amount of advertising is zero. As $M \to \infty$, we would expect the increased advertising to cause the attendance to remain nearly constant, driving the modified decay rate to zero. Thus the factor should be zero when the advertising budget is infinite. There are many ways to model such a factor; we choose a simple rational function:

$$\frac{1}{1+\tilde{\gamma}M/\tilde{S}} = \frac{\tilde{S}}{\tilde{\gamma}M+\tilde{S}},$$

where $\tilde{\gamma}$ is a constant describing the effectiveness of M in reducing the decay rate. This idea of a 'saturation effect', where the added utility of an additional incremental resource decreases with increasing resources, appears in such varied areas as insurance pricing (Bowers *et al.*, 1986) and population dynamics (Murray, 1993).

It should be noted that in general M will depend on $\tilde{t}$, as advertising is usually concentrated in the weeks surrounding the release. In addition, dollars may be pulled from a poorly performing movie and spent on one for which the rate of return may be higher. This assumption would make M a time-dependent variable whose evolution equation would have to be modeled as well. However, for simplicity in this paper we assume that M is a constant.

2.2.4 *Relaxing restriction #4: word of mouth.* Lastly we consider the effect of the audience reaction, which produces a similar perturbation of the decay rate. An important nuance here is that not only is $H_{\%}$, the percentage of viewers who dislike the film, important, but so is $\tilde{H}$, the total number of attendees to date who dislike the film. If the vast percentage of people who see the film hate it, but only 10 people saw the picture, this has minimal impact on revenue. Similarly, if 10 000 people hate the film, but this reflects only 5% of the people who viewed the film, then this also may have minimal impact on the film's box-office performance. We combine the two effects multiplicatively to produce a factor $\tilde{\beta} H_{\%} \tilde{H}$, where $\tilde{\beta}$ is a constant describing the effectiveness of $\tilde{H}$ in increasing the decay rate.

We choose to combine the effects from #3 and #4 additively, which produces our final evolution equation for $\tilde{A}$:

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tilde{t}} = -\frac{\tilde{\alpha}_A}{D(1+\epsilon)}\left[\frac{\tilde{S}}{\tilde{S}+\tilde{\gamma}M} + \tilde{\beta}\tilde{H}H_{\%}\right]\tilde{A}.$$

At this stage it is convenient to make a change of variables using (5) in the above to obtain

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tilde{t}} = -\frac{\tilde{\alpha}_A}{D(1+\epsilon)}\left[\frac{\tilde{S}}{\tilde{S}+\tilde{\gamma}M} + \frac{\tilde{\beta}}{P}\tilde{G}H_{\%}^2\right]\tilde{A}. \tag{7}$$

2.3 *Decay rate of screens*

Next we consider the evolution over time of the number of screens on which a film is shown. In contrast to the attendance rate, there are instances where the number of screens does not simply decrease over time.

In a 'platform release', a movie is initially released on a small number of screens. If as time goes on there is enough positive word of mouth to justify it, the number of screens on which the movie is playing slowly increases. Clearly there must be some point at which a maximum number of screens is reached, and then the number of screens decays with time after that. More typical is the 'wide release' pattern. In a wide release, the number of screens on which the movie is presented is initially large and does not increase, but usually simply decreases with time.

These two kinds of 'release pattern' behavior must be described by the differential equation for $\tilde{S}$. First, if the number of screens is small (below some critical threshold S_*) and the weekly attendance per screen average, $\tilde{A}$, is high, then the number of screens on which the movie is playing should increase. This is the platform release pattern.

Furthermore, if $\tilde{S} > S_*$, then the number of screens should strictly decay, no matter the value of $\tilde{A}$. This is the 'wide release' pattern.

The following evolution equation has the requisite properties:

$$\frac{\mathrm{d}\tilde{S}}{\mathrm{d}\tilde{t}} = -\tilde{\alpha}_S \left(\tilde{S} - S_* \frac{\tilde{A}}{A_{\max}} \right). \tag{8}$$

Here $A_{\max}$ corresponds to the capacity of each screen location; hence the fraction $\tilde{A}/A_{\max}$ can never be larger than 1. (Note that for simplicity we assume that all screens have the same capacity.) Thus if $\tilde{S} > S_*$, $\mathrm{d}\tilde{S}/\mathrm{d}\tilde{t} < 0$ and the number of screens decreases. However, if $\tilde{S} < S_*$, then for large values of $\tilde{A}$ it is possible for $\mathrm{d}\tilde{S}/\mathrm{d}\tilde{t} > 0$ and the number of screens to increase.

We note from (8) that we have hypothesized that $\tilde{S}$ depends only on $\tilde{A}$, and in a relatively straightforward manner. We have omitted considerations of advertising, etc., as in (7). This is because for the exhibitor, the income per screen is the main factor used to decide whether to continue showing a film, and it is that income, not the number of screens, which is directly affected by advertising, reviews, word of mouth, etc., as modeled directly in (7).

In addition, by modeling the evolution of $\tilde{S}$ in this way, we are neglecting the effect of the rental contract, which keeps the value of $\tilde{S}$ nearly fixed for the first few weeks after a film's release. We expect that this defect in the model will be felt most keenly for poorly received films, where our model would (correctly) direct exhibitors to pull a poorly performing film when to do so would violate their contract. We discuss this issue more in Section 7, as well as possible alternative models.

2.4 *Scaling the equations*

Our mathematical model is represented by (2), (7), and (8). We scale the problem by letting

$$S(t) = \frac{\tilde{S}(\tilde{t})}{S_*}, \qquad A(t) = \frac{\tilde{A}(\tilde{t})}{A_{\max}}, \qquad G(t) = \frac{\tilde{G}(\tilde{t})\tilde{\alpha}_S}{S_* A_{\max}}, \qquad t = \tilde{t}\tilde{\alpha}_S. \tag{9}$$

After substituting these scalings into (8) one obtains

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -(S - A). \tag{10}$$

Turning our attention to (2), if we again use information from (9) we have the following:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = SA, \qquad G(0) = 0. \tag{11}$$

Finally, making our substitutions into (7), we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\alpha \left(\frac{S}{S + \gamma} + \beta G \right) A, \tag{12}$$

where

$$\gamma = \frac{\tilde{\gamma} M}{S_*}, \qquad \alpha = \frac{\tilde{\alpha}_A}{\tilde{\alpha}_S} \frac{1}{D(1+\epsilon)}, \qquad \beta = \frac{\tilde{\beta} H_{\%}^2 S_* A_{\max}}{P \tilde{\alpha}_S}. \tag{13}$$

Note that we have constructed our system such that all the parameters lie in only one evolution equation—the one for A.

Our model is now given by (10)–(12). Lastly, we specify initial conditions for A and S:

$$A(0) = A_0, \qquad S(0) = S_0. \tag{14}$$

There are three unknown parameters: α, β and γ. α reflects the relative importance of screen decay rate versus the attendance decay rate. β incorporates the effect of word of mouth. γ is the effectiveness of the marketing and advertising campaigns.

2.5 *Some remarks on parameter estimation*

We conclude this section by characterizing the dependence of the various parameters in our model. They fall into several categories or groups:

(A) Parameters fixed for all movies, and hence can be estimated *a priori* from a database. These include P, the ticket price, $A_{\max}$, the maximum revenue that can be generated per screen, and S_*, the critical threshold beyond which the number of screens cannot increase.

(B) Parameters that depend on genre, cast, etc., which can also be estimated *a priori* from a database. These include $\tilde{\alpha}_A$, the natural decay rate for attendance, $\tilde{\alpha}_S$, the natural decay rate for screens, and $\tilde{\beta}$, the effect of word-of-mouth.

(C) Parameters that depend on each individual movie, and hence must be estimated during the movie's lifetime. M, the amount spent on advertising, should be known before or shortly after the movie's release, as will ϵ, the critical rating. D, the number of times an average moviegoer will see the film, $\tilde{\gamma}$, the effectiveness of advertising, and $H_{\%}$, the unfavorability proportion, can be estimated during the run of the movie.

The fact that these parameters must be estimated on the fly is not as detrimental to the model as it might first appear. Due to the initial contractual obligation, distributors are in essence locked in to a fixed run. It is only after the period ends (and several weeks' worth of data has been tabulated) that distributors must make decisions about where to spend advertising dollars most effectively, which contracts to renew, etc.

With an appropriately designed model, a simple minimization procedure implemented in a program such as Matlab can fit the solutions of (10)–(12) to the data, thus generating parameter estimates which can be used by the distributors for decision-making. It must be noted that such a first estimate would be necessarily crude due to the limited number of data points available. However, with each succeeding week the model could be refined incorporating new data as they become available. Sawhney & Eliashberg (1996) implement just such a refinement scheme in their probabilistic model. (For more details of how such a process could be carried out in practice, see Section 6.)

3. Asymptotic results

Equations (10)–(12) are a complicated system that cannot be solved analytically in closed form. However, in certain parameter regimes, we may obtain closed-form asymptotic results.

3.1 *One-parameter reductions*

3.1.1 $\alpha \to \infty$. We begin by focusing on the case where $\alpha \to \infty$. This corresponds to the case where $\tilde{\alpha}_A \gg \tilde{\alpha}_S$. This happens when the response time of the screens is very long; i.e. because of contractual or other restrictions, the number of screens remains relatively constant no matter how the movie is performing. It also corresponds to the case where the type of film is such that we expect attendance to decay quickly.

With $\alpha \to \infty$, there is no balance on the right-hand side of (12) other than $A = 0$. Therefore, there must be an initial layer during which the attendance decays rapidly. Since A and S are initially O(1), the $\mathrm{d}G/\mathrm{d}t$ term in (11) must also be O(1). Thus both G and t must be scaled in equal proportion and so we let

$$G(t) = \alpha^{-a} g(\tau), \qquad \tau = \alpha^a t, \quad a > 0. \tag{15}$$

Note that this corresponds to a small gross, since the attendance rate is dropping quickly. With this substitution, one can see from (10) that S evolves on a time scale slower than τ and hence on the τ scale is fixed at the constant initial value S_0.

Substituting this result and (15) into (11) and (12), we have the following:

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = S_0 A, \tag{16}$$

$$\alpha^a \frac{\mathrm{d}A}{\mathrm{d}\tau} = -\alpha(r_1 + \beta\alpha^{-a} g)A, \qquad r_1 = \frac{S_0}{S_0 + \gamma}, \tag{17}$$

from which we have that $a = 1$. Solving (16) and (17), we obtain, to leading order in α,

$$A(\tau) = A_0\,\mathrm{e}^{-r_1\tau}, \tag{18}$$

$$g(\tau) = \frac{S_0 A_0 (1 - \mathrm{e}^{-r_1\tau})}{r_1}. \tag{19}$$

From the form of (10)–(12) we know that the steady state for S has to be $S = 0$. Therefore, one could ask what happens on the long time scale t where S decays. Since $A(\tau = \infty) = 0$ by (18), there is no change to G on the t time scale.

3.1.2 $\alpha \to 0$. When $\alpha \to 0$, one can see from (12) that A evolves on a slower time scale, and hence on this time scale $A(t) = A_0$. Substituting this result into (10) yields

$$S(t) = A_0 + (S_0 - A_0)\,\mathrm{e}^{-t}, \tag{20}$$

where we have used (14).

On this time scale, G never approaches a constant because neither S nor A goes to zero, and hence we must introduce a long time scale. Again, since A and S are initially O(1) on this time scale, we see that the $\mathrm{d}G/\mathrm{d}t$ term in (11) must also be O(1). Therefore, both G and t must be scaled in equal proportion and so we again use the scaling in (15). Note that this corresponds to a large gross, which we would expect for a movie where the attendance decays very slowly.

Substituting (15) into (10), we obtain

$$\alpha^a \frac{\mathrm{d}S}{\mathrm{d}\tau} = -(S - A) \qquad \Longrightarrow \qquad S(\tau) = A(\tau). \tag{21}$$

Note that this is acceptable since $A_0 = S(t = \infty) = S(\tau = 0) = A(\tau = 0)$. Substituting (15) and (21) into (11) and (12), we have the following:

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = A^2, \tag{22}$$

$$\alpha^a \frac{\mathrm{d}A}{\mathrm{d}\tau} = -\alpha\left(\frac{S}{S+\gamma} + \beta\alpha^{-a} g\right)A, \tag{23}$$

from which we conclude that $a = 1/2$ for a dominant balance. Combining these equations yields

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\tau^2} = -2\beta g \frac{\mathrm{d}g}{\mathrm{d}\tau}, \qquad \frac{\mathrm{d}g}{\mathrm{d}\tau}(0) = A_0^2. \tag{24}$$

Equation (24) is a form of a *logistic equation*; in the Appendix we present a brief discussion of the equation and its solution in order to introduce notation we shall use throughout this paper.

From the Appendix it can be seen that (24) is just (54) with $G = g$, $G_0 = A_0^2$, $t = \tau$, $a = 0$, and $b = 2\beta$. Therefore, we may obtain our solutions from (55)–(57):

$$g_\pm = \pm A_0/\sqrt{\beta}, \qquad r_2 = 2A_0\sqrt{\beta}, \tag{25}$$

$$g(\tau) = g_+ \tanh\left(\frac{r_2\tau}{2}\right), \qquad A(\tau) = A_0 \operatorname{sech}\left(\frac{r_2\tau}{2}\right), \tag{26}$$

where in the last expression we have used (22).

3.1.3 $\beta \to \infty$. The case where β is very large corresponds to the case where the effect of bad word of mouth is quite strong, causing the movie to close very quickly. In this case there is no balance in (12), since the last term on the right-hand side is dominant. Therefore, we must either introduce a fast time scale (corresponding to the movie closing quickly) to make the left-hand side of (12) larger, scale G to make the βG term smaller, or both.

Since A and S are initially O(1), we see that the $\mathrm{d}G/\mathrm{d}t$ term in (11) must also be O(1). Therefore, both G and t must be scaled in equal proportion, so we let

$$G(t) = \beta^{-a} g(\tau), \qquad \tau = \beta^a t, \quad a > 0, \tag{27}$$

in (12) to obtain

$$\beta^a \frac{\mathrm{d}A}{\mathrm{d}\tau} = -\left(\frac{S}{S+\gamma} + \beta^{1-a} g\right)A, \tag{28}$$

from which we conclude that $a = 1/2$. Note that this implies that G is very small, corresponding to a film that closes quickly. Substituting (27) with $a = 1/2$ into (10), we see that S evolves on a slower time scale, and hence on the τ scale, the number of screens S is fixed at the constant initial value S_0. Again, this corresponds to a case where the demise of the film is so rapid that there is not time to reduce the number of screens.

Finally, making the substitutions into (28) and (11), we have the following:

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = S_0 A, \tag{29}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\tau^2} = S_0(-gA) = -g\,\frac{\mathrm{d}g}{\mathrm{d}\tau}, \qquad \frac{\mathrm{d}g}{\mathrm{d}\tau}(0) = S_0 A_0, \qquad g(0) = 0. \tag{30}$$

But (30) is just (54) with $G = g$, $G_0 = S_0 A_0$, $t = \tau$, $a = 0$, and $b = 1$. Therefore, we may obtain our solutions from (55)–(57):

$$g_\pm = \pm\sqrt{2S_0 A_0}, \qquad r_2 = g_+, \tag{31}$$

$$g(\tau) = g_+ \tanh\left(\frac{g_+\tau}{2}\right), \qquad A(\tau) = A_0\,\mathrm{sech}^2\left(\frac{g_+\tau}{2}\right), \tag{32}$$

where in the last expression we have used (29). Again we note that since $A(\tau = \infty) = 0$ by (32), the fact that S decays on the t time scale will not affect G.

3.2 $\beta \to 0$: *two-parameter reductions*

When $\beta \to 0$, some work will show that the resulting system is still not solvable in closed form. Therefore, we are forced to look at combinations of parameter regimes when $\beta \to 0$.

3.2.1 $\beta \to 0$, $\gamma \to 0$. When $\beta \to 0$ and $\gamma \to 0$, the effect of word of mouth is quite low, so the decay rate for A is basically the natural decay rate. In this case, the solution of (12) becomes, to leading order in β,

$$A(t) = A_0\,\mathrm{e}^{-\alpha t}. \tag{33}$$

Substituting (33) into (10) and solving subject to (14), we obtain

$$S(t) = \frac{A_0(\mathrm{e}^{-t} - \mathrm{e}^{-\alpha t})}{\alpha - 1} + S_0\,\mathrm{e}^{-t}, \qquad \alpha \neq 1. \tag{34}$$

Note that (34) does not hold in the case $\alpha = 1$, which is the case when the decay rates for the screens and the average is the same. In this case, we have

$$S(t) = A_0 t\,\mathrm{e}^{-t} + S_0\,\mathrm{e}^{-t}, \qquad \alpha = 1. \tag{35}$$

Substituting (33)–(35) into (11) yields

$$G(t) = \frac{A_0[1 - \mathrm{e}^{-(\alpha+1)t}]}{\alpha + 1}\left(S_0 + \frac{A_0}{\alpha - 1}\right) - \frac{A_0^2(1 - \mathrm{e}^{-2\alpha t})}{2\alpha(\alpha - 1)}, \qquad \alpha \neq 1, \tag{36}$$

$$G(t) = A_0^2\left[\frac{1}{4} - \left(\frac{1}{4} + \frac{t}{2}\right)\mathrm{e}^{-2t}\right] + \frac{A_0 S_0(1 - \mathrm{e}^{-2t})}{2}, \qquad \alpha = 1. \tag{37}$$

3.2.2 $\beta \to 0$, $\gamma \to \infty$. Now we look at the case where $\beta \to 0$ and $\gamma \to \infty$. This corresponds to the case where the effect of word of mouth is quite low, so the decay rate for A is very long. Thus on this time scale $A(t) = A_0$, so (20) holds in this case as well.

Since A and S are initially $O(1)$, we see that the $\mathrm{d}G/\mathrm{d}t$ term in (11) must also be $O(1)$. Therefore, both G and t must be scaled in equal proportion and so we again use the scaling in (27). Note that this implies that G is very large, corresponding to a film that does a lot of business because the average stays high. Substituting (27) into (10), we have the following:

$$\beta^a \frac{\mathrm{d}S}{\mathrm{d}\tau} = -(S - A) \qquad \Longrightarrow \qquad S(\tau) = A(\tau). \tag{38}$$

Equations (38) and (21) agree because both model the dynamics on long time scales. Substituting (27) and (38) into (11), we obtain

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = A^2. \tag{39}$$

Substituting (27) and (38) into (12), we obtain, to leading orders,

$$\beta^a \frac{\mathrm{d}A}{\mathrm{d}\tau} = -\alpha\left(\frac{A}{A+\gamma} + \beta^{1-a} g\right)A. \tag{40}$$

In order for the β terms to match, a must equal $1/2$. Therefore, (40) becomes, to leading order,

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = -\alpha\left(\frac{A}{\gamma\beta^{-1/2}} + g\right)A. \tag{41}$$

If $\gamma\beta^{-1/2} = O(1)$, then the system (39) and (41) cannot be solved in closed form. Therefore, to obtain analytical results, we have two choices: the case where $\gamma\beta^{-1/2} \to \infty$ or $\gamma\beta^{-1/2} \to 0$.

We consider the former case first. In this case, we may combine (39) and (41) to obtain

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\tau^2} = -2\alpha g A^2 = -2\alpha g \frac{\mathrm{d}g}{\mathrm{d}\tau}, \qquad \frac{\mathrm{d}g}{\mathrm{d}\tau}(0) = A_0^2, \qquad g(0) = 0. \tag{42}$$

But (39) and (42) are just (22) and (24) with $\beta = \alpha$. Therefore, our solutions are given by (25), (26):

$$g_\pm = \pm A_0/\sqrt{\alpha}, \qquad r_2 = 2A_0\sqrt{\alpha}, \tag{43}$$

$$g(\tau) = g_+ \tanh\left(\frac{r_2\tau}{2}\right), \qquad A(\tau) = A_0 \operatorname{sech}\left(\frac{r_2\tau}{2}\right). \tag{44}$$

Alternatively, if $\gamma\beta^{-1/2} \to 0$, then the perturbation expansion needs to be done in γ. Thus, analogous to the above, we let

$$G(t) = \gamma^a g(\tau), \qquad \tau = \gamma^{-a} t, \quad a > 0. \tag{45}$$

Again we have a large G and a long time scale, so (38) and (39) hold. Substituting (45) and (38) into (12), we obtain, to leading orders,

$$\gamma^{-a}\frac{\mathrm{d}A}{\mathrm{d}\tau} = -\alpha\left(\frac{A}{A+\gamma} + \beta\gamma^{a}g\right)A.$$

We expect a dominant balance between the left-hand side and the first term in the right-hand side, so we set $a = 1$. Hence we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = -\alpha A^2.$$

where the second term vanishes because of the case we are examining.

Continuing to solve, we have

$$A(\tau) = \frac{A_0}{1 + A_0\alpha\tau}, \qquad g(\tau) = \frac{A_0^2\tau}{1 + A_0\alpha\tau}. \tag{46}$$

Note that this is the only case where we have algebraic, rather than exponential, decay of A.

4. Numerical results: varying parameters

In this section we discuss how the numerical results are computed, as well as the variance of the solutions upon various parameters. Thus the sensitivity of the solution upon the parameters which must be estimated can be obtained.

To illustrate the effect of varying the various dimensionless parameters in the problem, we consider the specific case of *Runaway Bride*, which was the ninth highest-grossing film for the North American box office between April 1, 1999 and April 1, 2000 (Beilinson, 2000). The following dimensionless parameters were estimated using the box-office data:

$$\alpha = 0{\cdot}8350, \qquad \beta = 3{\cdot}9709, \qquad \gamma = 1{\cdot}3228. \tag{47}$$

In order to estimate the dimensionless parameters α, β, and γ, the data for each film were loaded into a Matlab file. The `lsqcurvefit` command was used to optimize the parameters by minimizing the residual between the data and Matlab's numerical solution of the system (10)–(12). The numerical solution was transformed back into dimensional variables before the optimization procedure so that the actual data (rather than scaled versions) could be fitted.

4.1 *Varying α*

Figure 1 shows the effect of varying α on A, the weekly revenue per screen. Throughout the paper, for graphing purposes the solutions have been transformed back into dimensional variables. That is, the time scale is weeks and the other axis has dimensions of screens or dollars.

Here the variation is about the 'ideal' value quoted in (47). Note that as α increases, the decay rate of A increases, as expected. From (13), we note that the movie-dependent

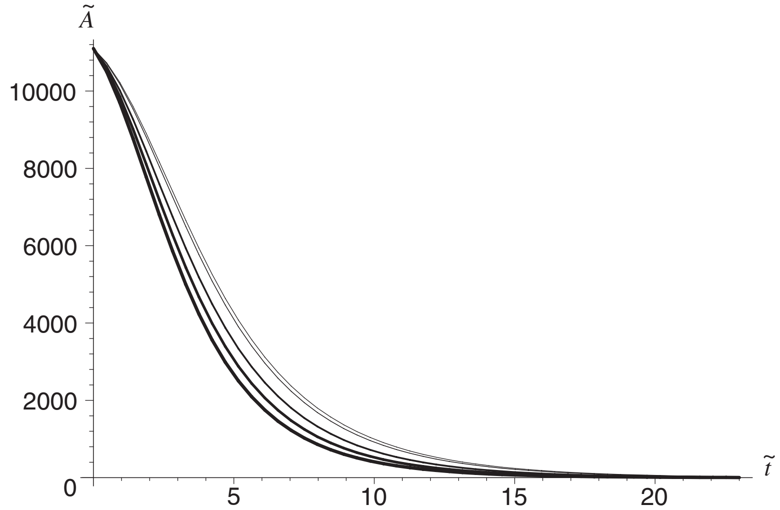

FIG. 1. Effect of varying α on $\tilde{A}$. In increasing order of thickness: $\alpha = 0{\cdot}95$, 1, 1·19, 1·38, and 1·57 times the value in (47).

parameters in α are D and ϵ. Thus, we see that if some fraction of the populace wants to see the film repeatedly, or the reviews are good, this will boost attendance over time.

However, we note that there is not much difference in the graphs. Since the vast proportion of moviegoers will not see the same film twice, D is bounded very close to 1, while ϵ has been constrained between $\pm 0{\cdot}05$. Therefore, though A may be sensitive to changes in α, its limited range means that as a practical matter α has little effect on the solution. Since S and G depend on α only through A, the variations in those graphs with α are similar.

4.2 *Varying β*

Figure 2 shows the effect of varying β on A. Note that as β increases, the decay rate of A increases, as expected. From (13), we note that the movie-dependent parameter in β is $H_{\%}$. Thus, if more people hate the movie, this will depress attendance over time.

Here there is a wide variation in the graphs due to the wide variation of β (nearly two orders of magnitude each way). Since $H_{\%}$ can vary widely between movies and β depends on its square, a wide variation in β is possible.

S and G depend on β only through A, but the wide variations in A cause wide variations in S and G, as shown in Figs 3 and 4.

4.3 *Varying γ*

Figure 5 shows the effect of varying γ on A. Note that as γ increases, the decay rate of A decreases, as expected. From (13), we note that the movie-dependent parameter in γ is M. Thus, if more advertising dollars are spent, this will increase attendance over time. Here

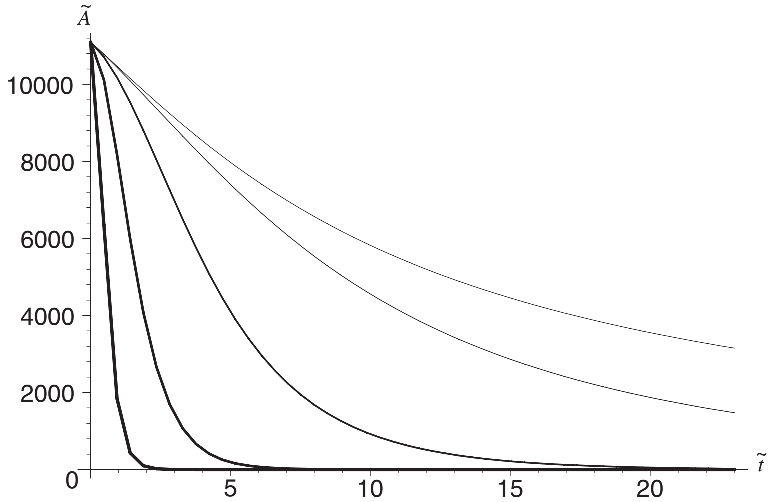

FIG. 2. Effect of varying β on $\tilde{A}$. In increasing order of thickness: $\beta = 1/81$, $1/9$, 1, 9, and 81 times the value in (47).

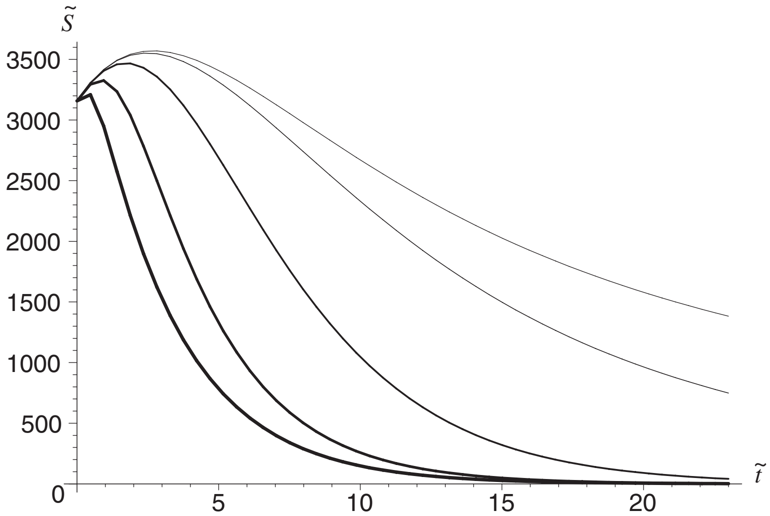

FIG. 3. Effect of varying β on $\tilde{S}$. In increasing order of thickness: $\beta = 1/81$, $1/9$, 1, 9, and 81 times the value in (47).

there is a relatively modest effect of $\tilde{A}$ on γ, since here γ varies only by around one order of magnitude each way.

Though S depends on γ only through A, we note from Fig. 6 that by increasing advertising dollars and hence attendance, the distributor can also increase the number of

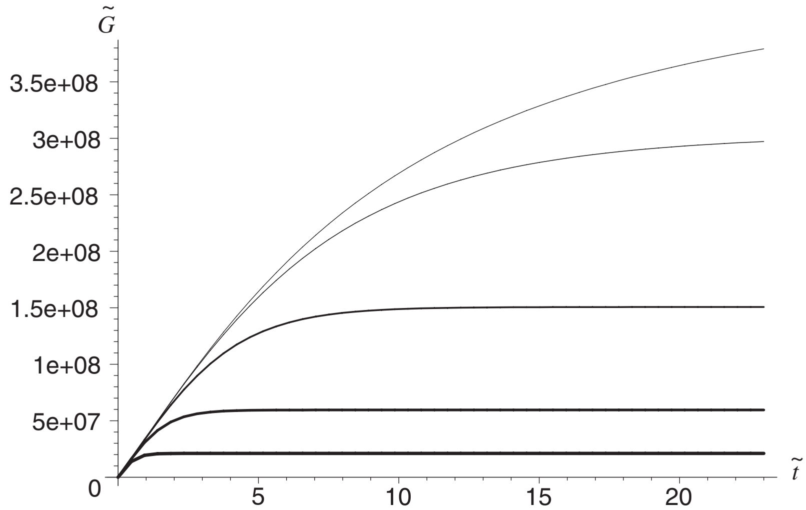

FIG. 4. Effect of varying β on $\tilde{G}$. In increasing order of thickness: $\beta = 1/81$, $1/9$, 1, 9, and 81 times the value in (47).

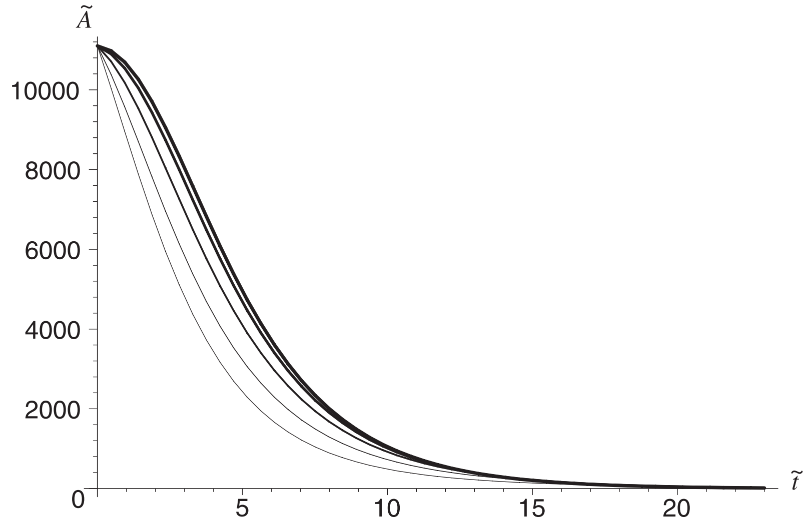

FIG. 5. Effect of varying γ on $\tilde{A}$. In increasing order of thickness: $\gamma = 1/9$, $1/3$, 1, 3, and 9 times the value in (47).

screens on which the film is showing, at least in the short term. The multiplicative effects of S and A can cause a rather substantial change in G, as shown in Fig. 7.

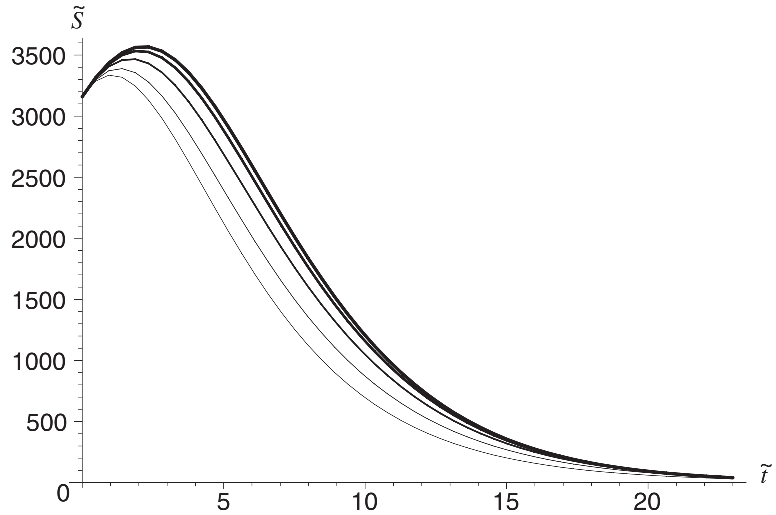

FIG. 6. Effect of varying γ on $\tilde{S}$. In increasing order of thickness: $\gamma = 1/9, 1/3, 1, 3$, and 9 times the value in (47).

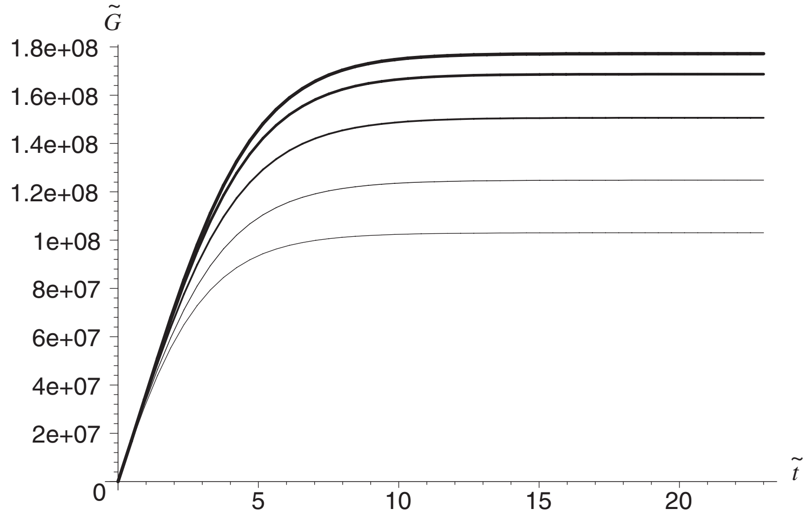

FIG. 7. Effect of varying γ on $\tilde{G}$. In increasing order of thickness: $\gamma = 1/9, 1/3, 1, 3$, and 9 times the value in (47).

4.4 *Varying S_**

We conclude this section with some remarks on the effect of S_*. We note from Fig. 8 that varying S_* (here by a factor of 2 each way) has very little effect on A, with increasing the number of screens increasing the attendance slightly. This small effect is expected, as S_*

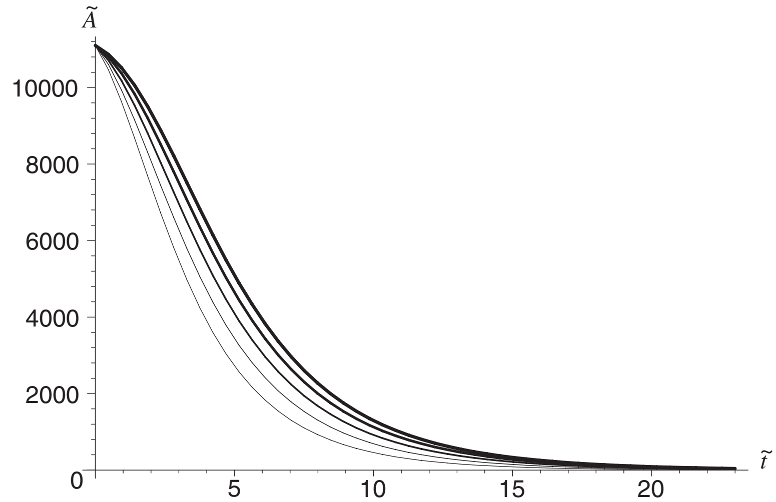

FIG. 8. Effect of varying S_* on $\tilde{A}$. In increasing order of thickness: $S_* = 2^{-1}, 2^{-1/2}, 1, 2^{1/2}$, and 2 times the value in (47).

is dictated by industry decision-making processes and does not reflect the choices of the public.

However, in Fig. 9 we see that varying S_* even by a small amount causes wide variations in the number of screens on which a film is shown. This sensitive dependence makes sense, as it is exactly the value of S_* that determines exactly how the studios distribute their films. This wide variation in S overcomes the small variation in A when the two are multiplied, thus producing moderate variation in grosses, as shown in Fig. 10.

5. Numerical results: comparison with data

In this section we discuss comparisons between actual box-office data and the results of numerical simulations using our model.

The 'weekend box-office estimates' announced in the mass media on Sundays are actually produced by the studios themselves, and are hence unreliable. The official, verified box-office numbers are usually available in the Tuesday edition of the *Los Angeles Times* as well as *Variety* and the *Hollywood Reporter*. Almost all of these media outlets use Exhibitor Relations, Inc. as the source of their data. A number of websites have developed which are devoted to maintaining archival databases of this box-office data; we chose Beilinson (2000) as our source. This website claims to contain box-office data for every film released in North America since 1997.

We looked at revenue data for the North American box office between April 1, 1999 and April 1, 2000. For this period we selected three films from separate parts of the revenue spectrum: *Runaway Bride* (9th of 442), *At First Sight* (67th of 442) and *Jawbreaker* (166th of 442). In the following sections we shall describe the individual features of the box-office behavior of these three films and provide numerical and graphical results which indicate

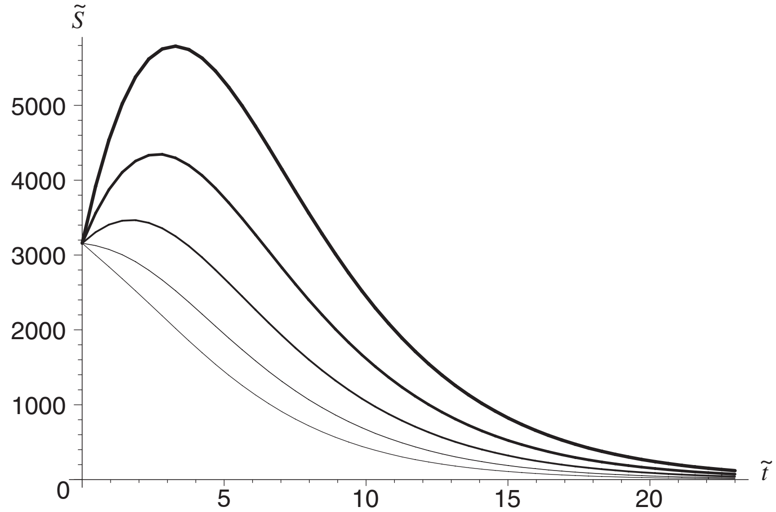

FIG. 9. Effect of varying S_* on $\tilde{S}$. In increasing order of thickness: $S_* = 2^{-1}, 2^{-1/2}, 1, 2^{1/2}$, and 2 times the value in (47).

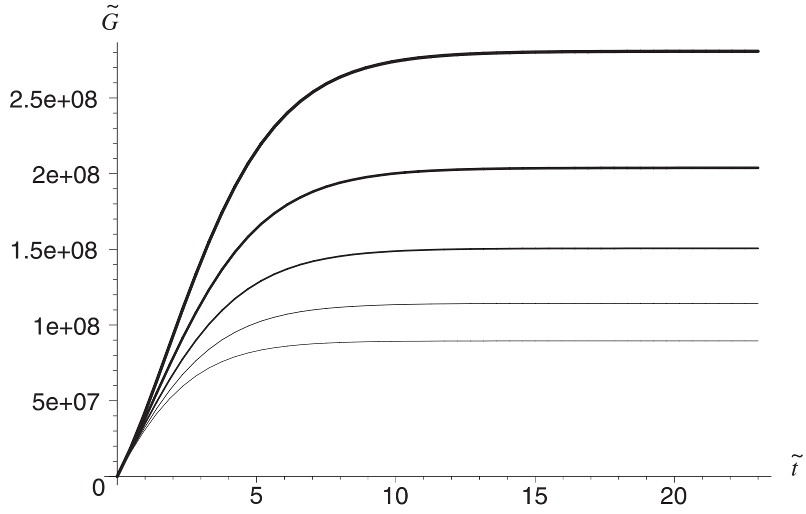

FIG. 10. Effect of varying S_* on $\tilde{G}$. In increasing order of thickness: $S_* = 2^{-1}, 2^{-1/2}, 1, 2^{1/2}$, and 2 times the value in (47).

that the mathematical model derived in this paper is flexible enough to encompass them all.

5.1 *Runaway Bride*

As one of the top ten grossing films of the period examined, *Runaway Bride* is a good example of a *blockbuster*. A blockbuster film is one which grosses an exceedingly large amount of money (customarily over 100 million dollars). There are usually only a few blockbusters per year. Blockbusters are hard to produce and almost impossible to predict (Rosen, 1981). In the last decade, the movie industry has modified its release pattern so that the studios have become more geared towards revenue from a few blockbuster movies. More recently, they have focused energy on increasing the number of blockbusters produced. In fact, in the period studied for this paper there were a record 21 movies which broke the 100 million dollar barrier, and four made over 200 million dollars (Beilinson, 2000). These movies are easily recognizable. They are released in a blizzard of a multimillion dollar marketing/advertising campaign consisting of fast-food marketing tie-ins, hit music videos, celebrity interviews, billboards, interactive websites and television commercials. They are incredibly expensive to produce and market, but can also generate huge amounts of revenue for the distributors (studios) and exhibitors.

The box-office behavior of a blockbuster is almost always geared around the opening weekend gross. The idea is that the release of the blockbuster becomes an 'event' and the public rushes to see the movie at the first available time (opening weekend). Thus blockbusters are released on a huge number of screens (2500–3000 screen openings are becoming commonplace) and generate a significant amount of gross in the first few days of release. In fact, since opening-weekend gross is used as a crude heuristic estimate of final gross, such figures are closely scrutinized for wide-release films to identify potential blockbusters.

After the initial release, the second weekend box-office gross is also important. If a film has a reduction in weekend gross of more than 25%, its chance at profitability may be in jeopardy and the film may be considered to be a disappointment. *Titanic*, which has the highest North American box-office gross of any film (Beilinson, 2000), earned that title in part because it was able to sustain weekend North American grosses of at least twenty million dollars for each of its first ten weeks. Another characteristic of a blockbuster's box-office gross is the length of time which it remains on the box-office charts. Most films disappear from the weekly box-office rankings within the first or second month. Blockbusters often remain on the box-office charts for half a year or more.

Once the dimensionless parameters in (47) have been computed, they can be translated back into dimensional parameters using (13). We again classify the parameters into groups as in Section 2.5:

$$\text{Group A: } P = 5{\cdot}08, \qquad A_{\max} = 22860, \qquad S_* = 8750.$$

The value for P comes from MPAA (1999). The value of $A_{\max}$ arises from assuming that on average, each screen has 250 seats, shows a film 18 times a week, and has an average ticket price given by P. The value of S_* is several times higher than the number of screens on which even the largest-release film would be shown. However, from (8) we note that S_* is the value beyond which the screens would not be increased *if every seat in every theatre were filled*. Since that never happens, an effective limit is reached for smaller values of $\tilde{A}/A_{\max}$, which implies a large value of S_*.

$$\text{Group B: } \tilde{\alpha}_S = \tfrac{1}{3},\ \tilde{\beta} = 10^{-5}.$$

The value for $\tilde{\alpha}_S$ (which is measured in inverse weeks) was estimated by noting that since there are usually three-week contractual obligations, the number of screens cannot decay any faster than that, even if $\tilde{A} = 0$.

We note that $\tilde{\beta}$ is simply a constant of proportionality to relate the strength of negative word-of-mouth, as expressed in $H_{\%}$. Since we do not have access to values of $H_{\%}$ (the data from test screenings is proprietary), for our purposes it is sufficient to provide a rough estimate that leads to reasonable $H_{\%}$ values (namely, values less than 1 and values that decrease with more successful movies). In practice, a distributor would do just the reverse: given a set of previously released films with values for $H_{\%}$ and β, the distributor could calculate $\tilde{\beta}$ for each genre.

Given the dimensionless parameters, how do we estimate the dimensional parameters? First, we see from (13) that we have all the components of the β equation except $H_{\%}$. Upon solving, we obtain

$$H_{\%} = 0{\cdot}0580, \tag{48}$$

a low number consistent with the movie's popularity.

For the γ equation, both M and $\tilde{\gamma}$ are unknown. In practice, a manager would know the value of M, and would want to estimate the effectiveness of the advertising through $\tilde{\gamma}$. If $\tilde{\gamma}$ is small, then more advertising may not be warranted. Also, if $\tilde{\gamma}$ has been estimated, the effect of changes in future advertising budgets may be ascertained. Unfortunately, we do not have the distributor's luxury of having hard and fast data for M. Instead, we noted from MPAA (1999) that the average studio film has marketing expenses of 24·53 million dollars for marketing and 6·52 million dollars for advertising. Since M is a *rate* at which advertising dollars are spent, we multiplied by $\tilde{\alpha}_S$ to obtain a characteristic rate of advertising dollars spent, given by

$$M = 1{\cdot}03 \times 10^7. \tag{49}$$

With the value of M given in (49), we obtain the following value for $\tilde{\gamma}$:

$$\tilde{\gamma} = 1{\cdot}12 \times 10^{-3}. \tag{50}$$

For the α equation, $\tilde{\alpha}_S$, D, and ϵ are unknown. We expect that upon the film's release, the reviews can be categorized and assigned an approximate numerical value of ϵ given their content. In addition, filmgoer surveys should provide a rough estimate of D. Then these values can be used to calculate $\tilde{\alpha}_S$ for each genre. For *Runaway Bride*, the reviews were mixed, but the movie was popular, so we let $\epsilon = 0$ and $D = 1{\cdot}02$. This led to the following calculated value for $\tilde{\alpha}_S$:

$$\tilde{\alpha}_S = 0{\cdot}284. \tag{51}$$

Figure 11 shows the prediction of the model for the gross receipts G of *Runaway Bride* using the parameters in (47). Note that we have data for nearly six months from the box-office charts, and so certainly in terms of length of run, the film qualifies as a blockbuster.

Figure 12 shows the prediction of the model for the number of screens S using the parameters in (47). We note that with the parameters used, the model predicts a growth in

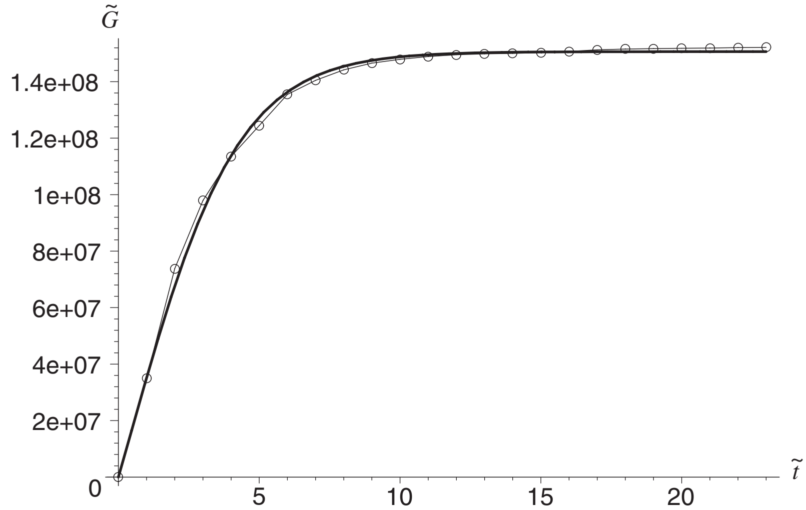

FIG. 11. Gross for *Runaway Bride*. Curve: prediction. Circles: actual data.

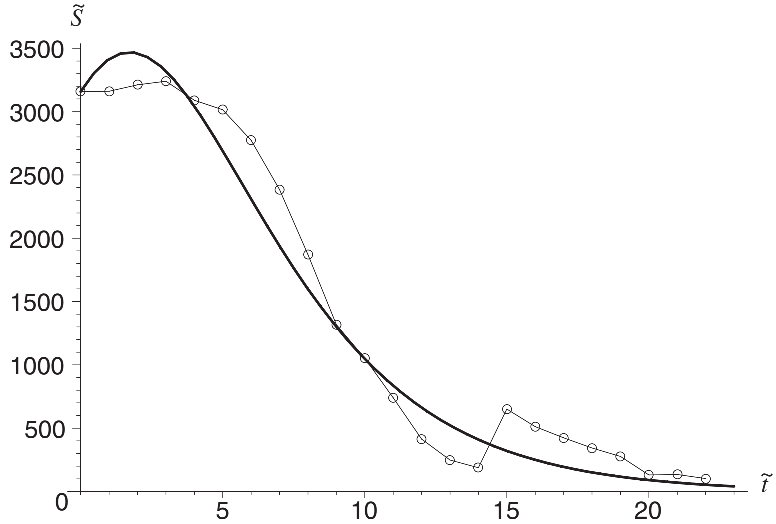

FIG. 12. Screens for *Runaway Bride*. Curve: prediction. Circles: actual data.

the number of screens, but overestimates the small uptick actually seen in week 4. However, the agreement is qualitatively good.

Figure 13 shows the prediction of the model for the per-screen revenue A using the parameters in (47). Note that the modeling errors in A and S cancel out when multiplied together, thus producing the good agreement for G in Fig. 11. Here we note that we are unable to model the uptick in attendance for the second week. This is because, unlike (10),

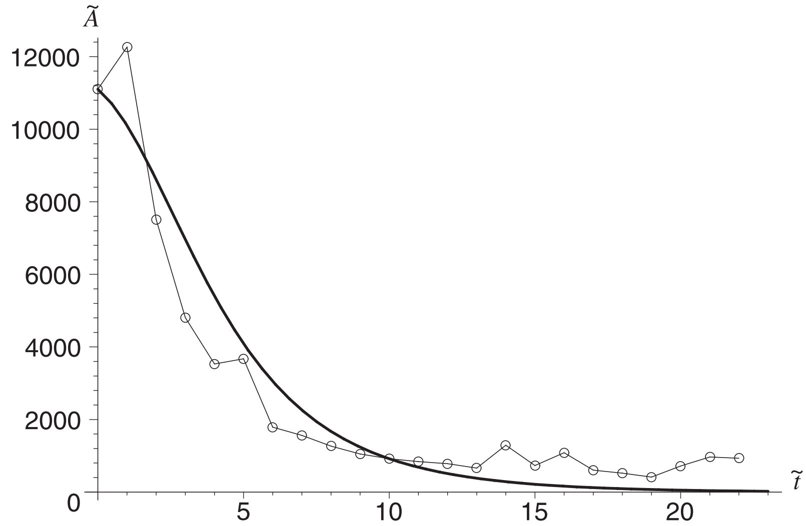

FIG. 13. Attendance for *Runaway Bride*. Curve: prediction. Circles: actual data.

which allows a growth in the number of screens, (12) does not allow a growth in attendance under any circumstances. Though this type of behavior is highly unusual (as will be seen later), any second-generation model should be able to capture this behavior.

Though occasionally seen in blockbusters, this rise in attendance is more indicative of a ‘sleeper’. Due to a small marketing budget or a low initial release, such films are often predicted to gross only a small amount. However, due to positive word of mouth, reviews, or awards, the films gain in attendance. A typical example of such a film is *The Blair Witch Project*. Such films are initially released on a modest number of screens, where audience reaction engenders a relatively high per-screen average. Following (10), the film is then released on more and more screens, leading to more people seeing the film. This causes positive feedback, which will peak a movie’s revenue later in the run than a traditional release.

5.2 *Jawbreaker*

Jawbreaker, which grossed 166th out of 442 movies surveyed, was one of the worst performers in the time period. Though it grossed more than 276 other movies, recall that the measurement is total gross receipts. The vast majority of the films rated below *Jawbreaker* were independent or foreign films screened in only a few locations which had no possibility of making large amounts of money. Thus the per screen average gross for *Jawbreaker* was among the worst we encountered.

Such an unsuccessful film is characterized as a *bomb* or a *dud*. These are generally films which the studio had supported with large production and marketing budgets in the full expectation that they would be blockbusters. Instead, the studios neither recoup their expenditures nor make a profit.

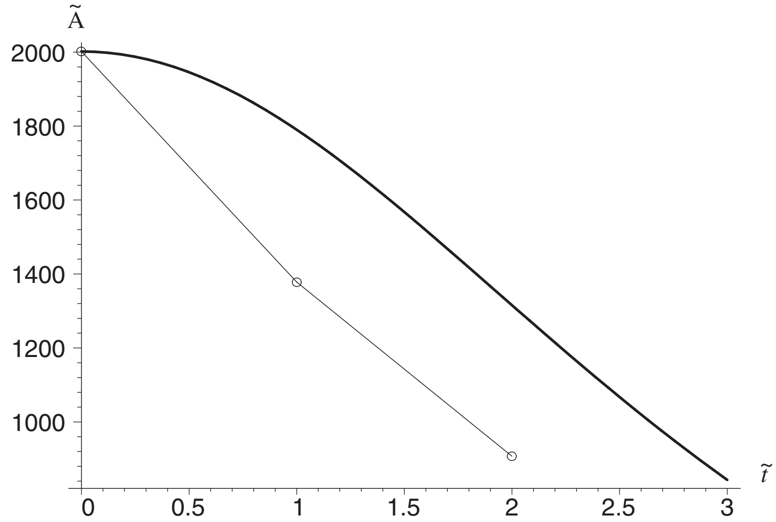

FIG. 14. Attendance for *Jawbreaker*. Curve: prediction. Circles: actual data.

The dimensionless parameters estimated to apply for this film are

$$\alpha = 0{\cdot}3505, \qquad \beta = 730{\cdot}3476, \qquad \gamma = 22{\cdot}6584. \tag{52}$$

The reviews for *Jawbreaker* were almost uniformly bad; hence we take the lowest possible value of ϵ: $\epsilon = -0{\cdot}05$. We also note that with such a poor-performing film, people are not apt to want to see it again, so we take $D = 1$. Since we do not know the exact value of the advertising, we again pick the average value given in (49). Following the same techniques as in Section 5.1, we obtain

$$H_{\%} = 0{\cdot}7863, \qquad \tilde{\alpha}_S = 0{\cdot}111, \qquad \tilde{\gamma} = 0{\cdot}0192.$$

Audiences did not seem to like the movie either, which corresponds to the extremely high value of $H_{\%}$. As discussed in Sections 4.1 and 4.2, note that the value of α does not change very much between a good film and a bad film, while the value of β changes dramatically—here by several orders of magnitude. Note that this puts us in the asymptotic case discussed in Section 3.1.3.

Note the higher value of $\tilde{\gamma}$, which would seem to indicate that advertising is more effective in this case than in that of a blockbuster like *Runaway Bride*. The key is to understand that in this case the advertising would act to *counteract* bad word of mouth, rather than enhance positive word of mouth, as in the case of a blockbuster. However, just because advertising may be effective does not mean it would be cost-efficient.

Figure 14 shows the prediction of the model for the per-screen revenue A for *Jawbreaker* using the parameters in (52). Note that, as is typical of bombs, the movie was pulled from theatres after only three weeks. This is due to the plummeting attendance rate, which is predicted well by our model.

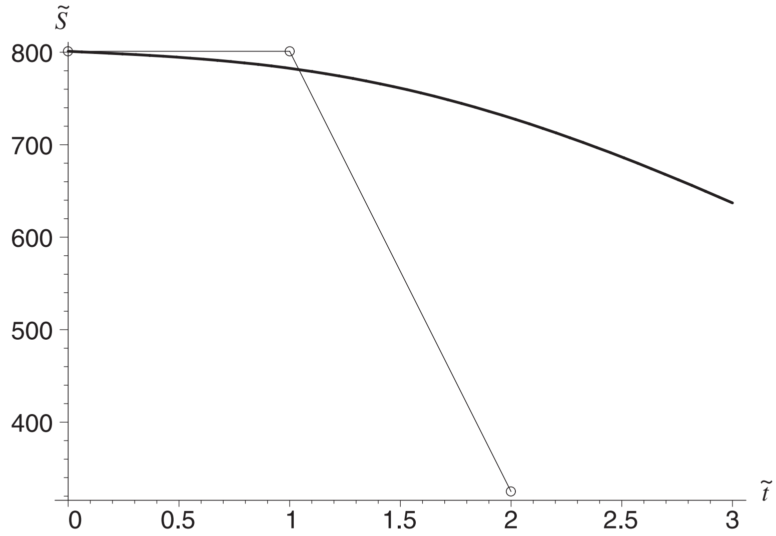

FIG. 15. Screens for *Jawbreaker*. Curve: prediction. Circles: actual data.

However, our model is not equipped to deal with precipitous declines in the number of screens, as shown in Fig. 15. By our assumptions in Section 2.1, our model can handle only exponential decay, even in the case of nonexistent attendance. Therefore, more refined models should include nonlinear effects to reduce the number of screens more quickly in the case of low attendance.

Because of our inability to model the screens well, even the small errors in the prediction for A will be magnified when predicting the gross. Thus, even in the optimum case the model does not predict the gross that well, as can be seen in Fig. 16.

5.3 *At First Sight*

We have now examined two movies from very different ends of the economic spectrum. In one case, the model works quite well, while in another, the agreement is more qualitative than quantitative. Thus we examine another film to see if we can identify patterns and characteristics of movies which our model simulates well.

We conclude by examining the film *At First Sight*, which grossed 67th out of 442 movies surveyed, putting it solidly in the middle of large-release big-budget films. The dimensionless parameters estimated to apply for this film are

$$\alpha = 1{\cdot}2688, \qquad \beta = 37{\cdot}0779, \qquad \gamma = 17{\cdot}0929. \tag{53}$$

The reviews for *At First Sight* were mixed-to-negative, so we take $\epsilon = -0{\cdot}01$. We take $D = 1$ again as well. Since we do not know the exact value of the advertising, we again pick the average value given in (49). Following the same techniques as in Section 5.1, we obtain

$$H_{\%} = 0{\cdot}1772, \qquad \tilde{\alpha}_S = 0{\cdot}419, \qquad \tilde{\gamma} = 0{\cdot}0145.$$

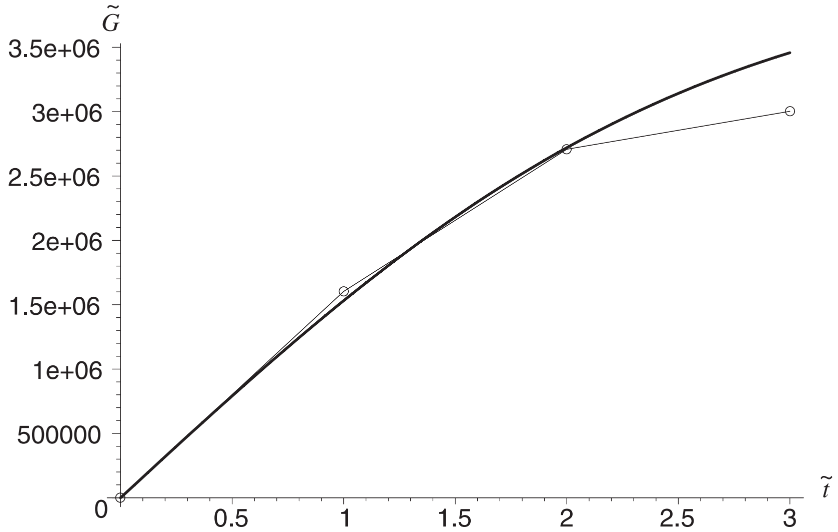

FIG. 16. Gross for *Jawbreaker*. Curve: prediction. Circles: actual data.

As this film lay in between *Runaway Bride* and *Jawbreaker* in terms of gross receipts, it is reasonable that the value of $H_{\%}$ should be between them, and indeed our estimate confirms this. Note that once again the value of β has changed dramatically with the moderate change in $H_{\%}$.

We also note that the value for $\tilde{\gamma}$ lies between the values for the other two films, as expected since the effect of negative word of mouth also lies between the previous two films.

Figure 17 shows the prediction of the model for the gross receipts G of *At First Sight* using the parameters in (53). Note that the film ran for ten weeks, which puts it midway between the two previously considered cases. Here we see that our model predicts the gross well for a moderately successful movie.

Figure 18 shows the prediction of the model for the per-screen revenue A using the parameters in (53). We note that though the data drop nearly exponentially, our model overestimates the attendance at the beginning and underestimates it at the end.

The reasons for the discrepancy can be discerned by examining Fig. 19, which shows the prediction of the model for the number of screens S using the parameters in (53). Using our model, the attendance data would indicate a reduction in the number of screens, as indicated by the line in the graph. However, we see that due to contractual obligations, the number of screens was fixed at an artificially high level for the first four weeks.

After the contract expired, there was a drastic reduction in the number of screens as the distributor rushed to cut its losses. Therefore, even though the revenue per screen A was respectable, since the number of screens was low, the weekly revenue was not high enough to justify an extended run. This can be further justified by noting from Fig. 13 that in the latter half of its run, the A figures for *Runaway Bride* were roughly the same as those for *At First Sight*, but *Runaway Bride* continued to run since the number of screens S was several

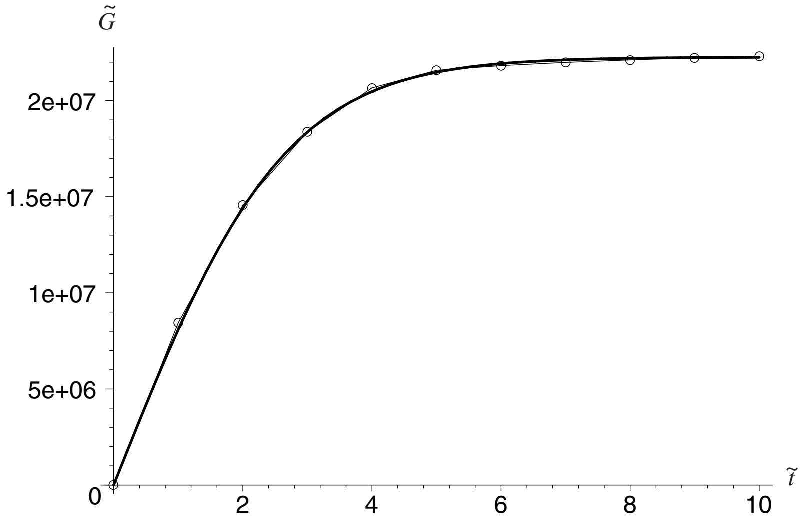

FIG. 17. Gross for *At First Sight*. Curve: prediction. Circles: actual data.

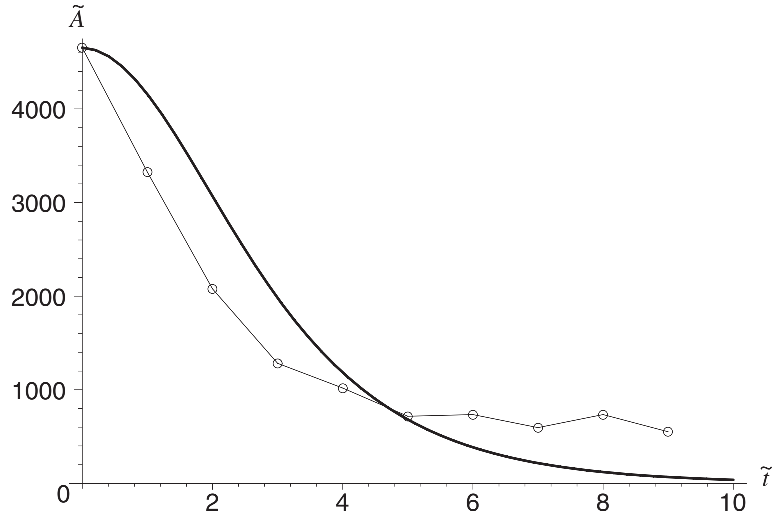

FIG. 18. Attendance for *At First Sight*. Curve: prediction. Circles: actual data.

times higher (see Fig. 12), and hence the weekly revenue was larger. In addition, *Runaway Bride* had established itself as a moneymaker over the beginning of its run, and hence it was a better bet to continue to produce substantial amounts of revenue.

This situation points out two facets of the movie business which any refined box-office model must incorporate. First, some modeling of the contractual limitation on S must be included, if in no other way than imposing $S = S_0$ for $t \in [0, t_{ex}]$, where t_{ex} is the

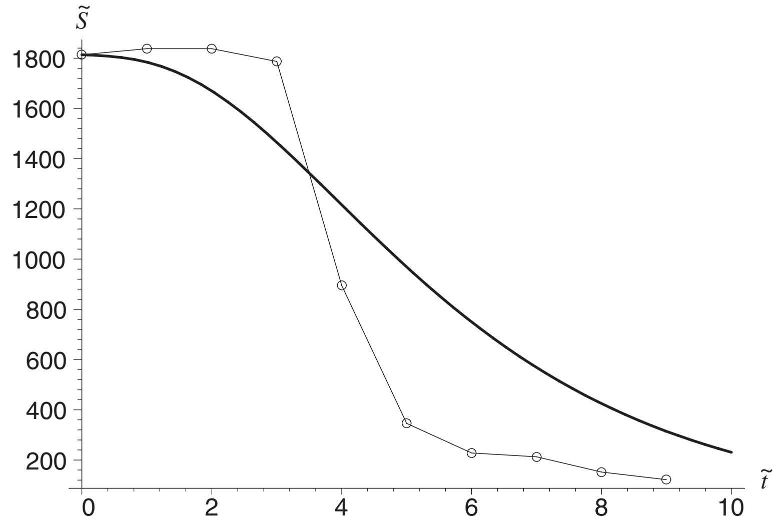

FIG. 19. Screens for *At First Sight*. Curve: prediction. Circles: actual data.

contract expiration date. This type of model would keep the system autonomous, but a more accurate model would then have to take into account the revenue history over those three weeks (perhaps through $G(t_{ex})$) to predict the behavior of the screens.

Second, we see that the differential equation (10) for S should probably depend on R directly, as it is considerations of weekly revenue that spur distributor and exhibitor decisions about the number of screens on which to show a film.

6. For the manager

A management model is useful only if it can be used in practice to help a company obtain its objectives. We present briefly a strategy for using the proposed model to do just that.

The first step is an application of the model to several years' of box office data. This will then yield the Group A parameters. Each of the movies must be classified according to genre and other characteristics in order to yield a picture of the Group B parameters for each category.

We note that with this background information in hand, certain weaknesses of the model can be minimized. In particular, if the model tends to under- or overestimate A and S for a particular genre in order to obtain a good estimate for G, this information can be used to adjust the values of A and S predicted by the model to real-life estimates.

Due to the three-week contract period, an exhibitor or distributor will always have three weeks' worth of data to use when performing a regression analysis. Once the parameters have been estimated, the manager can then easily input them into a Matlab ODE solver to extrapolate the model results to future weeks. Such an extrapolation will allow a manager not only to see how attendance and the gross will vary, but how the number of screens should be adjusted to enhance profits.

The same solver will also allow the manager to see how varying the parameters will affect the gross, as in Section 4. The manager can estimate from $\tilde{\gamma}$ the effectiveness a continued advertising campaign might have. The manager can see from $H_{\%}$ the effect of word of mouth.

Once another week's worth of data have been published, then the process can be repeated and refined. Trend lines will be particularly important in this iterative process. In particular, are the estimates for $H_{\%}$ increasing or decreasing? If this is week n, how do the predictions for S at week $n+1$ compare between this week's and last week's predictions? How well did last week's model predict this week's value for A? By examining these and other questions, managers will obtain the answers needed to adjust marketing campaigns and number of screens, which are basically the two variables at their disposal.

7. Conclusions

The filmmaking business is inherently a risky one. The budget for the entire film is spent before even one person purchases a ticket, and the product is one which the bulk of the customers will 'use' only once. The continued appearance of bombs (very unprofitable films) indicates that customer preferences are notoriously difficult to predict *a priori*. As De Vany & Walls (1999) discuss in their well-titled paper *'Uncertainty in the Movie Industry: Does Star Power Reduce the Terror of the Box Office?'*, almost any distribution of movie revenues is possible. Box-office performance has a probability distribution with infinite variance. However, a mathematical model that would predict the eventual gross of a film given initial performance data would be extremely helpful to distributors when they plan their future advertising budgets, negotiate for the number of screens on which to show a film, etc. This is precisely what we have attempted to present in this paper.

Previous authors (Sawhney & Eliashberg, 1996; De Vany & Walls, 1996, 1999; Neelamegham & Chintagunta, 1999; Eliashberg *et al.*, 2000) have used probabilistic models to estimate audience reaction and extrapolate total gross. In contrast, in this paper we use a deterministic model, and assume that the parameters in the model are probabilistic and hence must be estimated from the data. This eliminates debate over which distribution the parameters obey, since they are estimated for each film.

Our model consists of three coupled nonlinear ordinary differential equations for the unknowns: A, which is related to attendance, S, which is the number of screens, and G, the total gross. The parameters in the model fall into three main categories. 'Group A' parameters are fixed for all films, and are thus known *a priori*. 'Group B' parameters are assumed to be the same for all films which share the same characteristics (genre, production budget, etc.), and hence can be estimated with reasonable certainty from a large database of previously released films. 'Group C' parameters are movie-dependent and must be estimated from specific data for each film.

Just as one would expect from real-world experience, the key parameter in the mathematical system is $H_{\%}$, which models the percentage of people who do not like the film. The model depends on the square of this parameter, and thus even small changes in $H_{\%}$ can spell disaster or success for a film. The importance of this favorability percentage cannot be overestimated. Hollywood studios regularly 'test' their films and it has been reported that they will force directors to edit and re-edit their films until test audiences' reactions reach levels of at least 80% favorable (Puig, 2000).

In Section 3 we examined the model in several asymptotic limits. In each case we obtained results with agreed with our business interpretation. We considered the case where attendance drops off rapidly ($\alpha \to \infty$) or slowly ($\alpha \to 0$) compared to other decay rates in the problem. We were also able to model the case where there is extremely bad word of mouth ($\beta \to \infty$). When there was good word of mouth ($\beta \to 0$), the asymptotics required splitting into several different cases depending on the size of γ.

Since the full model cannot be solved analytically, we used Maple to solve the problem numerically, and in Section 4 we examined the sensitivity of the solution to changes in various parameters. Though the solution may be sensitive to changes in α, this parameter remains bounded in a very narrow range due to the nature of the dimensional parameters it includes. Since $\beta \propto H_{\%}^2$, the solutions vary greatly with changes in β. It is also β which must be estimated from any sort of predictive model, as it will provide guidance to the distributors and exhibitors for their next decision. We also demonstrated the variance of the solution with γ, which mainly models the effectiveness of advertising.

To validate the model, we compared it with films from three different parts of the revenue spectrum: a successful film (*Runaway Bride*), an unsuccessful film (*Jawbreaker*), and a moderately successful film (*At First Sight*). We found very good agreement with the first and third films, and qualitatively good agreement with the second. Thus, we have confidence that our model has the potential to predict box-office grosses, though clearly the more successful the film, the better the prediction. In some sense, this is acceptable, since studios can recognize poorly performing films already and cut their losses. It is the task of identifying which films will be successful that managers need to accomplish relatively quickly and the algorithm presented here assists in the completion of this task.

In the context of fitting the data, we found several deficits of this first model that would need to be addressed in any further work. First, (12) cannot model attendance rates that peak after a few weeks, as was the case with *Runaway Bride*.

Second, as evidenced in the case of *At First Sight*, due to the contractual obligations of the exhibition license the number of screens on which a film is initially shown for the first few weeks can be much higher than that predicted by the model. Such contractual obligations are standard in the business, and can account for the bulk of the gross for a moderately performing film. (Note from Fig. 19 that almost 90% of the gross occurred during the contract period.)

Thus, in a more refined model, (10) must include the effects of this contract period. If t_{con} is the end of the contract period, a possible replacement for (10) would be the following:

$$S(t) = S_0, \quad 0 \leqslant t \leqslant t_{\text{con}}, \qquad \frac{\mathrm{d}S}{\mathrm{d}t} = -(S - f(A)), \quad t \geqslant t_{\text{con}}.$$

Note that because of the contract period, exhibitors have additional information at their disposal—namely the data for A for the number of weeks in the contract. Thus, $f(A)$ may be modeled as the average of A over several weeks, the speed at which A has changed over the last few weeks, or some combination of the two.

Third, sound business practice dictates that the number of screens on which a film is exhibited must be related to the weekly revenue the film is producing, and this is indicated in the data for *At First Sight*. Equation (10) does not include such a direct dependence.

Despite these flaws, the agreement in Section 5 between actual box-office data and numerical prediction of grosses from the model shows that the mathematical modeling approach herein (namely a deterministic one) has great promise to describe and predict box-office grosses, and we expect the agreement only to improve with the refinements suggested above.

Future work would include the refinement of the model as a predictive tool. To accomplish this, each week's box office data would be incorporated into the analysis as they are published, thus improving the parameter estimates and hence the predictions. In addition, to be more useful, the model should incorporate the distributor's revenue based on the percentage of the gross they retain from the exhibitors. The model could then actually be used by managers on either side of the exhibitor/distributor divide to use the predictions to optimize their balance sheet.

However, the goal of this paper of presenting a differential-equation-based model of cinematic box-office dynamics has been clearly met. Without disparaging the probability- and statistics-based approaches of prior researchers, our approach appears to have produced output which mimics the data just as accurately, while using mathematical techniques to minimize the number of variables that must be estimated probabilistically.

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Appendix: the logistic equation

In order to simplify the analysis in various asymptotic limits, we derive a result we use repeatedly. Consider the following logistic equation:

$$\frac{\mathrm{d}^2G}{\mathrm{d}t^2} = -(a + bG)\frac{\mathrm{d}G}{\mathrm{d}t}, \qquad \frac{\mathrm{d}G}{\mathrm{d}t}(0) = G_0, \qquad G(0) = 0, \tag{54}$$

where a and b are constants. The solution of (54) is

$$G(t) = \frac{G_+G_-(1 - \mathrm{e}^{-r_2t})}{G_- - G_+\,\mathrm{e}^{-r_2t}}, \tag{55}$$

$$G_\pm = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} + \frac{2G_0}{b}}, \qquad r_2 = \frac{b(G_+ - G_-)}{2}. \tag{56}$$

We note from (56) that $G_- < 0$ and $G_+ > 0$. Thus $r_2 > 0$ and $G(\infty) = G_+$. We will also need $\mathrm{d}G/\mathrm{d}t$, which is given by

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{G_0\,\mathrm{e}^{-r_2t}(G_- - G_+)^2}{(G_- - G_+\,\mathrm{e}^{-r_2t})^2}. \tag{57}$$