Application of a Mickens finite-difference scheme to the cylindrical Bratu-Gelfand problem

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The boundary value problem $\Delta u + \lambda e^u = 0$ where $u = 0$ on the boundary is often referred to as "the Bratu problem." The Bratu problem with cylindrical radial operators, also known as the cylindrical Bratu-Gelfand problem, is considered here. It is a nonlinear eigenvalue problem with two known bifurcated solutions for $\lambda < \lambda_c$, no solutions for $\lambda > \lambda_c$ and a unique solution when $\lambda = \lambda_c$. Numerical solutions to the Bratu-Gelfand problem at the critical value of $\lambda_c = 2$ are computed using nonstandard finite-difference schemes known as Mickens finite differences. Comparison of numerical results obtained by solving the Bratu-Gelfand problem using a Mickens discretization with results obtained using standard finite differences for $\lambda < 2$ are given which illustrate the superiority of the nonstandard scheme.

Keywords: Mickens discretization, Bratu problem, Gelfand problem, nonstandard finite differences

This paper is dedicated to Professor Ronald Mickens to celebrate the occasion of his 60th birthday (February 7, 2003).

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1 Introduction

In this paper the results of applying a nonstandard finite-difference scheme to solve the cylindrical Bratu-Gelfand problem [10], a particular boundary value problem related to the classical Bratu problem [3], shall be presented. The Bratu problem is a nonlinear elliptical partial differential equation which appears in a number of applications, from the fuel ignition model found in thermal combustion theory [9] to the Chandrasekhar model for the expansion of the universe [8]. It is also a nonlinear eigenvalue problem that is often used

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as a benchmarking tool for numerical methods ([1], [2]) due to the bifurcated nature of the solution for $\lambda < \lambda_c$. In [11], Jacobsen and Schmitt provide an excellent summary of the significance and history of the Bratu problem. There, they consider the nature of solutions to a version of the problem generalized to more complicated operators in more dimensions that they call the Liouville-Bratu-Gelfand problem:

$$\begin{cases} r^{-\gamma}(r^{\alpha}|u'|^{\beta}u')' + \lambda e^{u} = 0, \qquad 0 < r < 1, \\ u > 0, \\ u'(0) = u(1) = 0, \end{cases} \tag{1}$$

where $\alpha \geq 0, \quad \gamma + 1 > \alpha, \quad \beta + 1 > 0$. The Bratu-Gelfand problem to be considered in this paper is the special case when $\alpha = 1, \quad \beta = 0, \quad \gamma = 1$.

$$\begin{cases} \frac{1}{r}(ru')' + \lambda e^{u} = 0, \qquad 0 < r < 1, \\ u > 0, \\ u'(0) = u(1) = 0. \end{cases} \tag{2}$$

The assumption has been made that $u = u(r)$ so that the other derivatives in the Laplacian can be ignored. One way to solve boundary value problems is to discretize the derivatives using finite-differences and then use Newton's Method to solve the resulting system of non-linear equations [2]. Nonstandard finite-difference schemes popularized by Professor Ronald Mickens ([18], [14], [17]) are numerical methods for solving differential equations which often have reduced (sometimes zero!) discretization errors. In previous work ([4],[5],[6]) I have shown the usefulness of applying a particular Mickens-type finite-difference scheme to boundary value problems in cylindrical coordinates that contain the expression $r\frac{du}{dr}$. The standard forward-difference discretization would be

$$r\frac{du}{dr} \approx r_k \frac{u_{k+1} - u_k}{r_{k+1} - r_k}. \tag{3}$$

However, the following nonstandard finite-difference scheme has been shown [6] to be a superior method, especially for singular problems where $r \to 0$.

$$r\frac{du}{dr} \approx \frac{u_{k+1} - u_k}{\log(r_{k+1}) - \log(r_k)}. \tag{4}$$

In section 2 of this paper the exact solution of the Bratu-Gelfand problem will be presented. In addition, the bifurcated nature of the solution shall be discussed. In section 3, the details of how to compute numerical solutions and some comments on using nonstandard finite differences shall be presented. Numerical solutions generated by the Mickens discretization (4) and the standard discretization (3) will be compared to the exact solution. The paper will conclude with some overall comments and observations based on the numerical results presented in the previous sections.

2 The Bratu-Gelfand Problem and its Exact Solution

The Bratu-Gelfand problem can also be written as

$$u''(r) + \frac{1}{r}u'(r) + \lambda e^{u(r)} = 0 \qquad 0 \le r \le R,$$
$$\text{with } u(0) < \infty \quad \text{and} \quad u(R) = 0.$$

The exact solution to (2) is given in [20] and is

$$u(r;\lambda) = \ln\left[\frac{b}{\left(1 + \frac{\lambda b}{8}r^2\right)^2}\right], \tag{5}$$

where b is given by

$$b = \frac{32}{\lambda^2 R^4}\left(1 - \frac{\lambda R^2}{4} \pm \sqrt{1 - \frac{\lambda R^2}{2}}\right). \tag{6}$$

Clearly there are only solutions when $\lambda \le \frac{2}{R^2}$. When $R = 1$ and more specifity about the inner boundary condition is given (i.e. $u'(0) = 0$) equations (5) and (6) can be combined to write down the solution to (2) as

$$u(r;\lambda) = \ln\left[\frac{\frac{32}{\lambda^2}\left\{1 - \frac{\lambda}{4} \pm \sqrt{1 - \frac{\lambda}{2}}\right\}}{\left(1 + \frac{4r^2}{\lambda}\left\{1 - \frac{\lambda}{4} \pm \sqrt{1 - \frac{\lambda}{2}}\right\}\right)^2}\right]. \tag{7}$$

The above expression in (7) has two values for every value of $0 < \lambda < 2$. For example, Figure 1 depicts the bifurcated behavior of the solution by depicting the two solutions for $\lambda = 1$ in relation to the unique solution obtained when $\lambda = 2$. The solution obtained from taking the positive square root in (7) shall be denoted as as $u_+(r;1)$ and $u_-(r;1)$ as the solution obtained when taking the negative square root in (7).

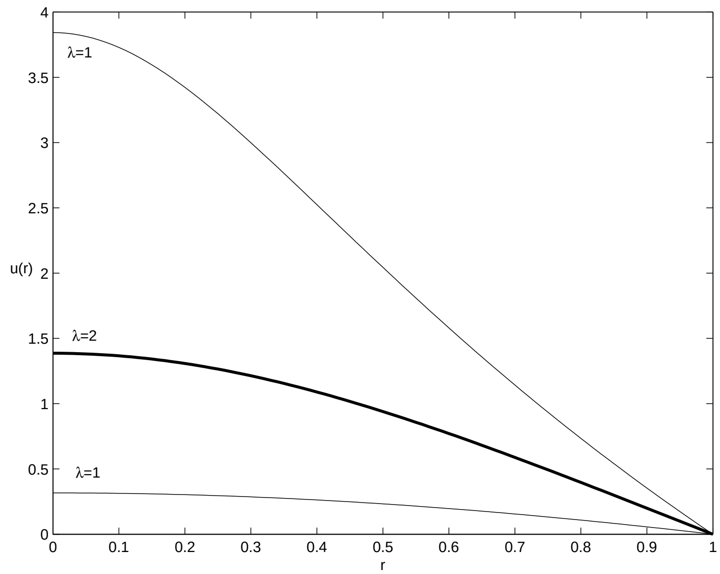


Figure 1: Exact solutions to the Bratu-Gelfand problem when $\lambda = 1$ (bifurcated) and $\lambda = 2$ (unique)

The exact form of the upper curve in Figure 1 is given by

$$u_+(r;1) = \ln \left[\frac{24 + 16\sqrt{2}}{\left(1 + r^2(3 + 2\sqrt{2}\right)^2} \right] \tag{8}$$

and the exact form of the lower curve is given by

$$u_-(r;1) = \ln \left[\frac{24 - 16\sqrt{2}}{\left(1 + r^2(3 - 2\sqrt{2}\right)^2} \right]. \tag{9}$$

The maximum value $||u||_\infty$ of both curves occurs at $r = 0$, and $u_+(0;1) = \ln(4) + \ln(6 + 4\sqrt{2}) = 3.84218871$ and $u_-(0;1) = \ln(4) + \ln(6 - 4\sqrt{2}) = 0.31669436$.

Another way to illustrate the bifurcated nature of the solution is to graph the maximum value of $u(r)$ on $0 \leq r \leq 1$ versus λ, as shown in Figure 2. This also clearly shows the "turning point" in the solution at the critical value of $\lambda_c = 2$.

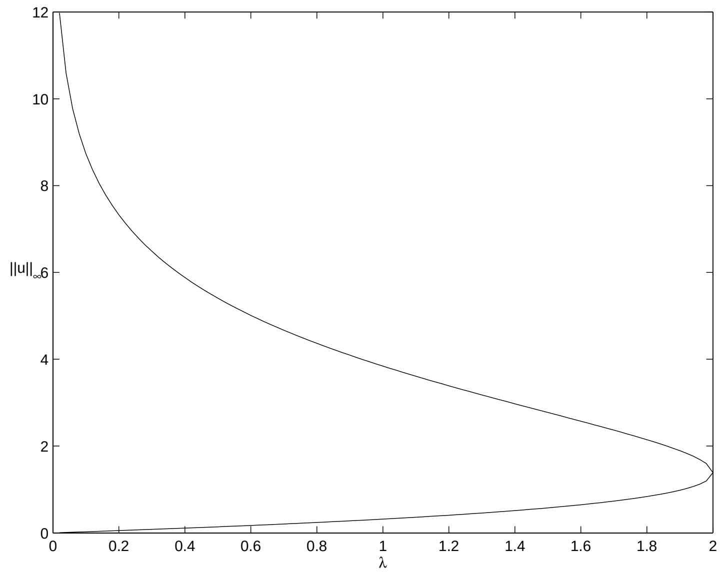


Figure 2: Maximum value of $u(r)$ versus λ depicting the turning point at $\lambda = 2$

The single-valued version of (7) that occurs when $\lambda = 2$ is astonishingly simple:

$$u(r) = \ln\left[\frac{4}{(1+r^2)^2}\right] = \ln(4) - 2\ln(1+r^2). \tag{10}$$

The graph of this function (10) is depicted in Figure 3. It is the exact solution to (2) and clearly obeys the boundary conditions $u(1) = 0$ and $u'(0) = 0$. Note also that its maximum value occurs at $r = 0$ and is exactly $\ln(4) = 1.38629436...$.

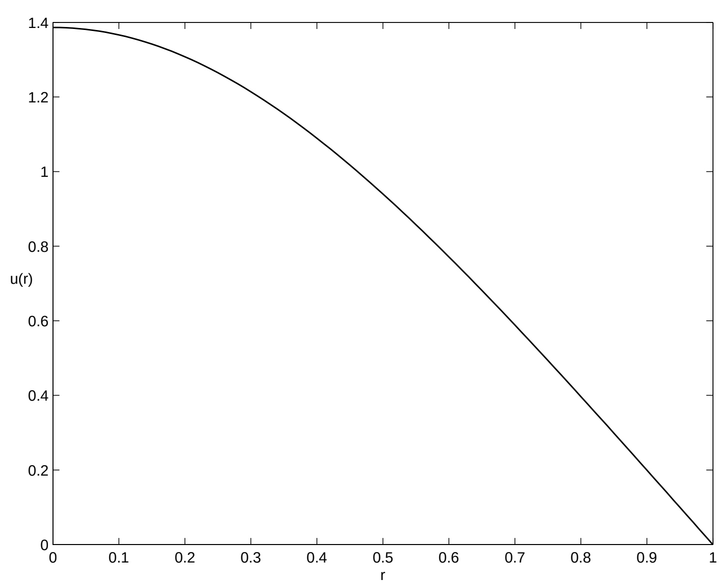


Figure 3: Exact solution of the Bratu-Gelfand problem when $\lambda = 2$

3 Numerical Method

Two different methods were used to compute numerical solutions to the Bratu-Gelfand problem (2) in order to compare them. Both methods involve forming discrete versions of the boundary value problem by approximating the derivatives and boundary conditions and solving the resulting system of nonlinear difference equations numerically. Exactly which method to use to approximate differential equations using difference equations on a discrete grid is a topic that has been extensively addressed by Ronald Mickens of Clark Atlanta University ([15], [16]). It is these methods that I refer to as "Mickens finite differences" or "Mickens discretizations."

3.1 A Primer on Mickens differences

A Mickens discretization of the derivative has the general form

$$\frac{du}{dr} \approx \frac{u_{k+1} - u_k}{\phi(h)}. \tag{11}$$

where h is the grid separation parameter and $\phi = h + o(h)$. The symbols u_k and u_{k+1} are the values of $u(r)$ at consecutive locations on the discrete grid. It is the form of the "denominator function" $\phi(h)$ which determines that the discretization will be part of a nonstandard finite difference scheme. In [12] some examples of denominator functions which have the same desired qualities are provided and reproduced below in (12). Obviously, the first denominator function in the list below would result in the standard forward-difference formula for the derivative.

$$\phi(h) = \begin{cases} h, \\ \sin(h), \\ e^h - 1, \\ 1 - e^{-h}, \\ \sinh(h), \\ \ln(1+h), \\ \dfrac{1 - e^{-\tau h}}{\tau}, \\ \vdots \end{cases} \tag{12}$$

In the limit as $h \to 0$, using any of the above sample denominator function results in the familiar forward-difference definition of the derivative. However, for finite h, which is, of course the typical computational practice, the denominator function will produce a discrete derivative that is an approximation to the standard derivative. The nature of the approximation to the derivative will depend on the size of h and the nonlinearity of ϕ.

There is another way in which a discretization of a derivative can produce a nonstandard discretization separate from the choice of denominator function. This involves using "nonlocal" terms in discretizations. For example, in spherical coordinates the operator $r^2 \frac{d}{dr}$ often appears. The standard discretization for this would be

$$r^2 \frac{du}{dr} \approx r_{k+1/2}^2 \frac{u_{k+1} - u_k}{r_{k+1} - r_k}, \tag{13}$$

where the r^2 terms are evaluated in between grid points. However, the following finite-difference scheme

$$r^2 \frac{du}{dr} \approx r_k r_{k+1} \frac{u_{k+1} - u_k}{r_{k+1} - r_k}, \tag{14}$$

is an example of a Mickens discretization which has been shown to have superior numerical behavior [6]. In (14) r^2 is being discretized as not r_k^2 or r_{k+1}^2 but as a surprising hybrid, $r_k r_{k+1}$. In [13] and elsewhere, Mickens provides examples of other, even more surprising non-local discretizations which lead to improved (sometimes even exact) accuracy over standard discretizations. The author encourages others to investigate using Mickens finite differences when attempting to numerically solve differential equations using the finite-difference approximation method.

3.2 Numerical Solutions of the Bratu-Gelfand Problem

The first step in the numerical solution is to discretize the domain of the problem. The grid chosen was $\{r_j\}_{j=0}^N$ on the interval $0 \le r \le 1$ where

$$0 = r_0 < r_1 < r_2 < \ldots < r_j < \ldots < r_N = 1. \tag{15}$$

For a uniform grid, the grid separation parameter h is constant and $h = 1/N$ with $r_k = 0 + kh$ for $k = 0$ to N. Using the standard finite-difference scheme (3) the discrete version of the Bratu-Gelfand problem (2) will be

$$\frac{1}{r_j}\left(r_{j+1/2}\frac{u_{j+1} - u_j}{r_{j+1} - r_j} - r_{j-1/2}\frac{u_j - u_{j-1}}{r_j - r_{j-1}} \right) + \lambda e^{u_j} = 0. \tag{16}$$

The nonstandard version finite-difference scheme (4) for (2) will be

$$\frac{1}{r_j}\left(\frac{u_{j+1} - u_j}{\log(r_{j+1}/r_j)} - \frac{u_j - u_{j-1}}{\log(r_j/r_{j-1})} \right) + \lambda e^{u_j} = 0. \tag{17}$$

Note: For the nonstandard scheme r_0 must be positive, i.e. $0 < r_0 << 1$. A simple discrete version of the inner boundary condition $u'(0) = 0$ is

$$\frac{u_1 - u_0}{r_1 - r_0} = 0 \Rightarrow u_1 = u_0. \tag{18}$$

Another, more accurate, version of the inner boundary condition is that the flux (i.e. ru' must be zero at the "first" grid point, which when substituted into (16) leads to the following equation at $j = 0$ using standard differencing

$$\frac{1}{r_0}\left(r_{1/2}\frac{u_1 - u_0}{r_1 - r_0} \right) + \lambda e^{u_0} = 0. \tag{19}$$

Using the nonstandard difference method (17) the discrete version of the inner boundary condition is

$$\frac{1}{r_0}\left(\frac{u_1 - u_0}{\log(r_1/r_0)} \right) + \lambda e^{u_0} = 0. \tag{20}$$

The discrete version of the outer boundary condition $u(1) = 0$ is

$$u_N = 0. \tag{21}$$

When $0 < \lambda < 2$ the system of nonlinear equations due to the standard discretization ((16),(19),(21)) and the system due to the Mickens discretization ((17),(20),(21)) are each solved very easily using Newton's Method. Computations are conducted using the exact solution $u(r; 2)$ (10) as an initial guess, with a tolerance of 10^{-8}.

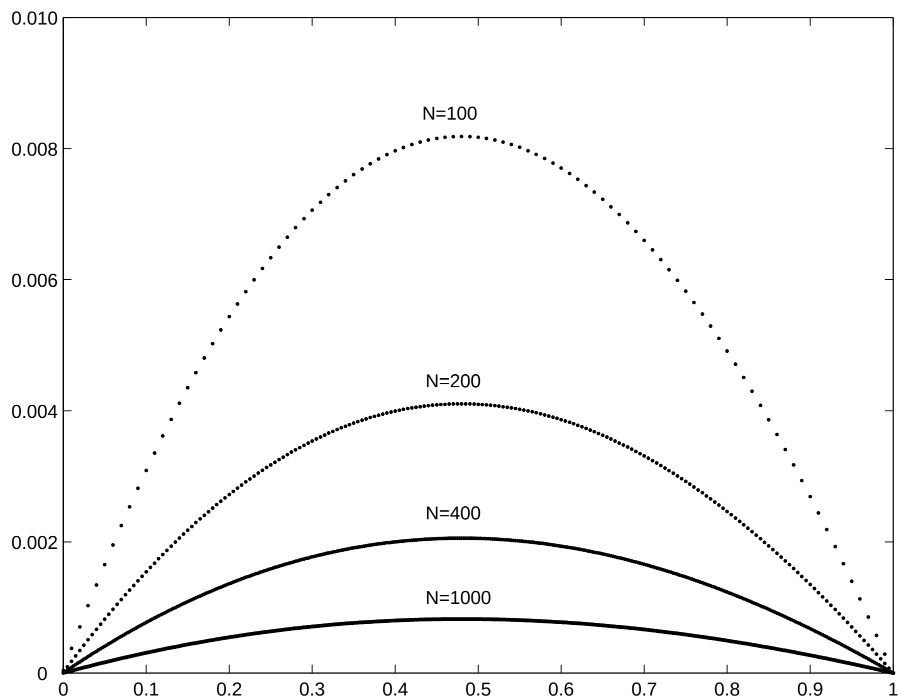
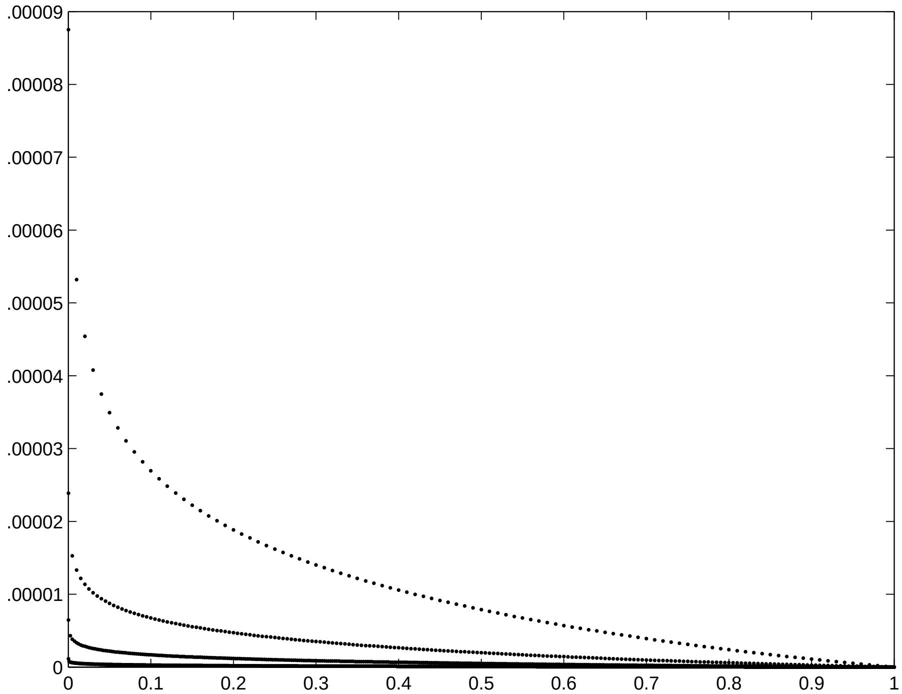
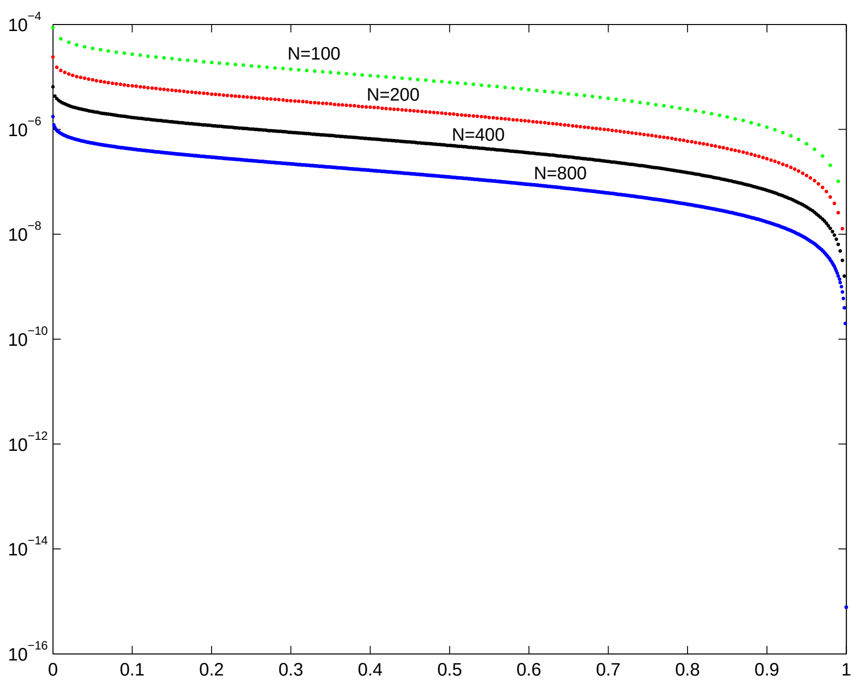


Figure 4: Numerical error versus r for standard method when $\lambda = 1$

The numerical errors generated by the two competing methods for $\lambda = 1$ and for various values of increasing N are given in Figure 4 and Figure 5. Notice the completely different quantitative and qualitative nature of the graphs. The second graph in Figure 5 illuminates the error behavior of the nonstandard method by using a semilog scale. The smallest maximum error in Figure 4 (the N=1000 curve) is greater than the largest maximum error in Figure 5 (the N=100 curve). Clearly the solution produced by the Mickens scheme has superior accuracy over the one generated using standard finite differences when $\lambda = 1$.



(a) abs(error) versus r



(b) log(error) versus r

Figure 5: Numerical error versus r using Mickens differences when $\lambda = 1$

At the turning point $\lambda = 2$ the system of equations generated by using the standard finite difference scheme refuses to converge. This is not unexpected since it is widely known that numerically computing solutions at or near the turning point is difficult using standard methods. However, the Mickens finite-difference method has no problem generating numerical solution at this critical value of the parameter λ.

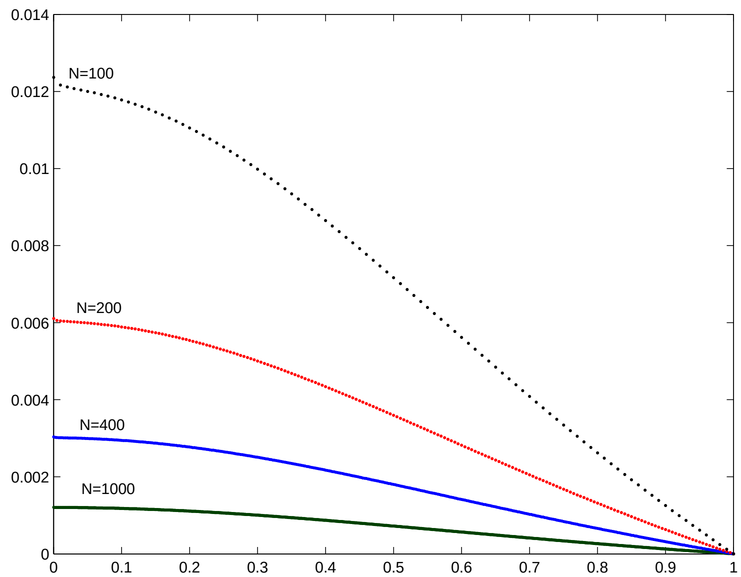


Figure 6: Numerical error using Mickens finite differences with $N = 100, 200, 400$ and 1000 for $\lambda = 2$

The numerical results of solving the Bratu-Gelfand problem at the critical value of $\lambda = 2$ are depicted in Figure 6. This shows that the error is greatest at $r = 0$ but that the error over the entire domain $0 \le r \le 1$ clearly goes to zero as the number of grid points N increases.

4 Conclusion

Mickens differences again illustrate their superior properties to standard finite differences in the example given here of solving the cylindrical Bratu-Gelfand problem numerically. This is true not only at the subcritical values of the parameter $\lambda < 2$ but even at the critical value of $\lambda = 2$ where standard finite differences fail to produce a convergent solution. Future work will consider the application of Mickens differences to the Bratu-Gelfand problem in spherical coordinates. In spherical coordinates the solution has an infinite number of turning points [19] which are quite challenging for most numerical methods to capture. In other work Mickens differences have been applied successfully to the one-dimensional, planar Bratu problem [7]. The planar Bratu problem has a very similar structure to the cylindrical Bratu-Gelfand problem considered here (a critical value of λ beyond which solutions don't exist and before which two solutions exist) and again Mickens differences produce superior results. The author welcomes correspondence with suggestions for other interesting problems to which Mickens differences could be applied.

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