Senior **C**olloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am

 `http://sites.oxy.edu/ron/math/400/f20/`

Class 1: **Tuesday August 25**

TITLE Dimensional Analysis

CURRENT READING Logan, Section 1.1 (pp. 1-8); Holmes, Section 1.2 (pp. 4-6)

NEXT READING Logan, Section 1.1, (pp. 8-29);

SUMMARY

This week we will be introduced to the first thing to think about when solving a problem in applied mathematics: what are the units (and how can I get rid of them?)

DEFINITION: **dimensional analysis**

The analysis of the physical dimensions of the variables and parameters in an equation.

The Pi Theorem

"Given a physical law that gives a relation among a certain number of *dimensioned* quantities, then there is an equivalent law that can be expressed as a relation among certain **dimensionless** quantities" (Logan 5).

The basic idea is that a physical law written as $f(x_1, x_2, x_3, \ldots, x_n) = 0$ can be written as $F(\pi_1, \pi_2, \ldots, \pi_m) = 0$ where x_i are dimensioned quantities (i.e. have units) while the π_i are combinations of the dimensioned quantities in such a way that the π_i are dimensionless quantities.

What are the primary dimensions?

Generally, we can take any physical dimensioned quantity and write it as some combination of primary or fundamental dimensions. There are generally considered seven fundamental dimensions, but in this class the ones we will most often use are: mass, length, time. These are usually denoted M, L and T and have the units kg, m and sec respectively.

NOTATION: $[\![\cdot]\!]$

Given a physical quantity q, the fundamental dimensions of q are denoted $[\![q]\!]$. When q is dimensionless, $[\![q]\!] = 1$. Therefor, for variables m l and t representing mass, length and time, $[\![m]\!] =$ M, $[\![l]\!] =$ L and $[\![t]\!] =$ T.

$\boxed{\text{EXAMPLE}}$

(Logan Example 1.1) Consider Taylor's law that relates the energy E released in an atomic explosion that depends on time t, the radius of the fireball r and density ρ, so that we assume there is a physical law $g(t, r, \rho, E) = 0$.
We'll use dimensional analysis to show what form the function g must take.

$$f\left(\frac{r^5\rho}{t^2E}\right) = 0$$

is a dimensionless version of Taylor's law.

(STEP 1): Show that $[\![\frac{r^5\rho}{t^2E}]\!] = 1$.

(STEP 2): Derive an expression for how the radius r varies in terms of time t

Physical quantity	Expression	Dimensional formula
Area	length × breadth	$[L^2]$
Density	mass / volume	$[ML^{-3}]$
Acceleration	velocity / time	$[LT^{-2}]$
Momentum	mass × velocity	$[MLT^{-1}]$
Force	mass × acceleration	$[MLT^{-2}]$
Work	force × distance	$[ML^2T^{-2}]$
Power	work / time	$[ML^2T^{-3}]$
Energy	work	$[ML^2T^{-2}]$
Impulse	force × time	$[MLT^{-1}]$
Radius of gyration	distance	$[L]$
Pressure	force / area	$[ML^{-1}T^{-2}]$
Surface tension	force / length	$[MT^{-2}]$
Frequency	1 / time period	$[T^{-1}]$
Tension	force	$[MLT^{-2}]$
Moment of force (or torque)	force × distance	$[ML^2T^{-2}]$
Angular velocity	angular displacement / time	$[T^{-1}]$
Stress	force / area	$[ML^{-1}T^{-2}]$
Heat	energy	$[ML^2T^{-2}]$
Heat capacity	heat energy/ temperature	$[ML^2T^{-2}K^{-1}]$

Table 1.1: Dimensions of some common physical quantities

Exercise

(Logan, Example 1.2) Find an expression for the maximum height h of a projectile thrown vertically in a gravitation field, assuming h will depend on the acceleration due to gravity g, the mass m of the ball and the velocity v (ignoring air resistance). Thus we assume a physical law $f(m, g, v, h) = 0$.

(STEP 1): Assume a dimensionless quantity Π can be formed from m, g, v, and h so that
$\Pi = m^\alpha g^\beta v^\gamma h^\delta$

(STEP 2): Show that $h = C\dfrac{v^2}{g}$

$\boxed{\text{GROUPWORK}}$

(Logan Example 1.3) $F = f(\rho, A, v)$ where A is cross-sectional area, v is speed and ρ is density. The force F of air resistance is related to these quantities in some way. Can you determine what it is? Use dimensional analysis!