Senior Colloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am
 `http://sites.oxy.edu/ron/math/400/f20/`

Class 9: Tuesday October 20

TITLE Introduction to Boundary Layer Theory & Asymptotic Matching

CURRENT READING Logan, §3.3 (pp. 179-185); Holmes, §2.5-2.6 (pp. 69-84); Witelski, §7.1-7.3 (pp. 147-158)

SUMMARY
This week we will be introduced to the concept of boundary layers, i.e. a solution to a boundary value problem which is only valid during certain regions of the independent variable.

BOUNDARY VALUE PROBLEM
A boundary value problem is very similar to an initial value problem except the known information about the unknown solution of the ODE is at different values of the independent variable.

$$\epsilon \frac{d^2y}{dx^2} + (1+\epsilon)\frac{dy}{dx} + y = 0, \text{ where } \epsilon \ll 1 \text{ and } \quad 0 < x < 1 \text{ with } y(0) = 0, \quad y(1) = 1. \tag{1}$$

We'll assume the usual regular perturbation solution of the form

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \dots \tag{2}$$

then we will produce a series of differential equations (with BOUNDARY conditions) of various orders in epsilon which look like...
The $\mathcal{O}(1)$ equation is

$$\frac{dy_0}{dx} + y_0 = 0, \quad y_0(0) = 0, \quad y_0(1) = 1 \tag{3}$$

The $\mathcal{O}(\epsilon)$ equation is

$$\frac{dy_1}{dx} + y_1 = -y_0'' - y_0', \quad y_1(0) = 0, \quad y_1(1) = 0 \tag{4}$$

EXAMPLE
Let's solve these boundary value problems and see what happens. $y_0(x) = Ae^{-x}$ is the homogeneous solution of (3). What happens when you solve for A by satisfying the boundary conditions?

So, this means that the form of the solution assuming a regular perturbation for $y(x)$ given in (2) will not work and we're looking at a **singular** perturbation problem and will have to figure out something else to approximate the solution.

We also could have seen this was a singular problem from the fact that setting $\epsilon = 0$ changes the ODE from a 2nd order problem to a 1st order problem.

Let's look again at the exact problem given in Equation (1)

$$\epsilon y'' + (1+\epsilon)y' + y = 0 \tag{5}$$

Notice that it has the form $ay'' + by' + cy = 0$ where a, b and c are constant coefficients which depend on ϵ. Assuming the ansatz of $y = e^{rx}$ you should be able show that the r satisfies the following equation

$$r = \frac{-(1+\epsilon) \pm (1-\epsilon)}{2\epsilon} \tag{6}$$

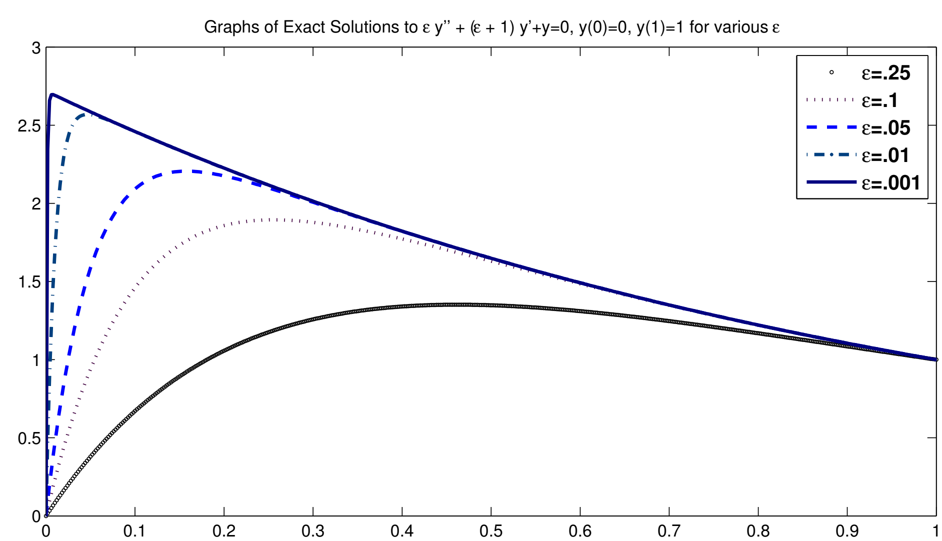
and so $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ will be the exact solution to (5).

Exercise

Confirm the exact solution to (5) by obtaining the values of r and solving for c_1 and c_2 by using the boundary conditions $y(0) = 0$ and $y(1) = 1$.

The exact solution is graphed below for $\epsilon = 0.1$

$$y(x) = \frac{1}{e^{-1} - e^{-1/\epsilon}}(e^{-x} - e^{-x/\epsilon}). \tag{7}$$



It is also found here: `https://www.desmos.com/calculator/qlmzlmvyng`

Boundary Layer

Near the $x = 0$ boundary of the interval $0 \leq x \leq 1$ the exact solution $y(x)$ given in Equation (7) changes very rapidly, in a very thin layer which appears to be related to ϵ. This area of rapid change is called a **boundary layer**. Many real-world physical problems like fluid flow possess solutions which exhibit boundary layers.

We obtain the solution of a problem with a boundary layer by splitting the problem into two: an inner approximate solution $y_{inner}(x)$ and an outer approximate solution $y_{outer}(x)$. Each piece of the problem has a region of validity.

Our goal is to obtain a function $y_{uniform}(x)$ which is uniformly valid over the entire region the ODE is defined over, and whose error or residual goes to zero in the limit as $\epsilon \to 0^+$.

The Outer Problem

The outer solution is easier to find because it is valid when ϵ is ignored, i.e. in the range where $\mathcal{O}(\epsilon) < x \leq 1$, so you can let $\epsilon = 0$ in the original problem given in (1)

$$y'_{outer} + y_{outer} = 0, \quad y_{outer}(1) = 1 \tag{8}$$

Notice the inner boundary condition near $x = 0$ is ignored.
Solving the IVP in (8) gives us the solution $y_{outer}(x) = e^{1-x}$.

Exercise

You should check that the outer solution $y_{outer}(x) = e^{1-x}$ satisfies the IVP given in Equation (8)

The Inner Problem
Recall that our given boundary value problem is

$$\epsilon y'' + (1+\epsilon)y' + y = 0, \qquad y(0) = 0, \quad y(1) = 1$$

For now we will assume that the width of the boundary layer is given as $\mathcal{O}(\epsilon)$ and we will re-scale the independent variable x by the width of the boundary layer. (We obtain the boundary layer by using dominant balancing but we will return to that topic next week. For now we can assume we know that $\delta(\epsilon) = \epsilon$.)

$$\xi = \frac{x}{\delta(\epsilon)} \text{ and } Y(\xi) = y(x) = y(\xi\delta(\epsilon)) \tag{9}$$

and plug in the new variables in (9) into the original equation in (1) produces

$$\frac{\epsilon}{\delta(\epsilon)^2}\frac{d^2Y}{d\xi^2} + \frac{(1+\epsilon)}{\delta(\epsilon)}\frac{dY}{d\xi} + Y(\xi) = 0 \tag{10}$$

Knowing the scaling $\delta(\epsilon) = \epsilon$ and plugging back into (10) leads to the inner problem

$$Y'' + Y' + \epsilon Y' + \epsilon Y = 0, \quad Y(0) = 0 \tag{11}$$

which is an ODE that can be approximated using regular perturbation, so we set $\epsilon = 0$ and consider the leading order problem $Y'' + Y' = 0$, which has the solution $Y(\xi) = A + Be^{-\xi}$ but since this is the inner solution, it should satisfy the inner boundary condition of $y = 0$ at $x = 0$ which means that $Y = 0$ when $\xi = 0$ so that $B = -A$ and the inner solution has the form $Y(\xi) = A(1 - e^{-\xi})$.

EXAMPLE

Let's show that the leading order solution to $Y'' + Y' + \epsilon Y' + \epsilon Y = 0, \quad Y(0) = 0$ is $Y(\xi) = A(1 - e^{-\xi})$.

If we want to convert between x variables and ξ variables we know that $\xi = \dfrac{x}{\epsilon}$ and $x = \xi\epsilon$
To summarize, we know have

$$\begin{aligned} y_{inner}(x) &= A(1 - e^{-x/\epsilon}), \text{ when } 0 \le x \le \mathcal{O}(\epsilon) \\ y_{outer}(x) &= e^{1-x}, \text{ when } \mathcal{O}(\epsilon) < x \le 1 \end{aligned}$$

NOTE: we still have an unknown constant in our expression for the inner solution. We need to find out A in order to get a complete solution. The process of finding the value of the constant A involves **asymptotic matching**.

Asymptotic Matching

In order to find the unknown constant in the inner solution we need a matching condition. It turns out that this is

$$\lim_{x\to 0^+} y_{outer}(x) = \lim_{\xi\to\infty} y_{inner}(\xi) = M = y_{BLC} \tag{12}$$

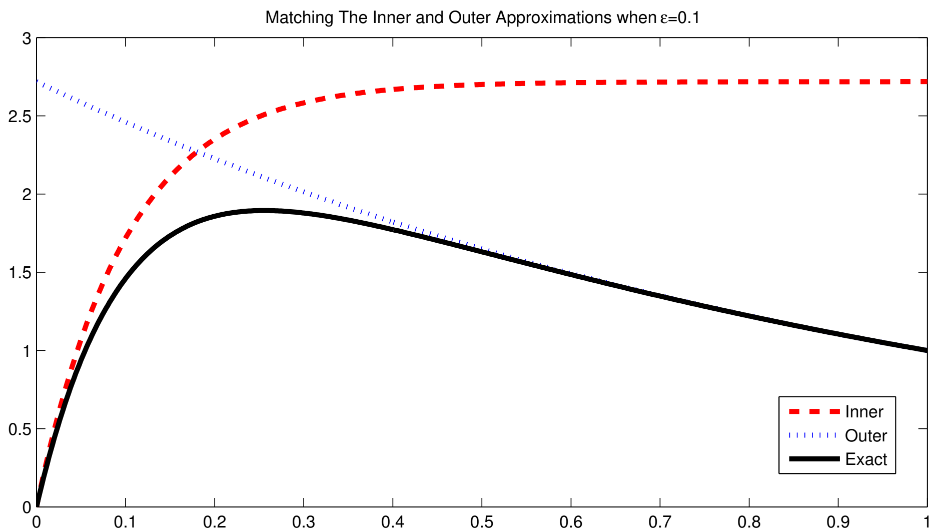
where M is the matched value equal to the value of both limits. Logan calls this the **boundary layer constant** and deotes it to be y_{BLC}. What this means is that the value as you go "in" from the outer solution ($x \to 0+$) and as you go out from the inner solution ($\xi \to \infty$) the inner and outer solutions "match" each other.

If we practice asymptotic matching for our y_{inner} and y_{outer} solutions in our problem above, we will see that $y_{BLC} = e$.

Exercise

Use the matching condition $\lim_{x\to 0^+} y_{outer}(x) = \lim_{\xi\to\infty} Y(\xi)$ to confirm the value for the unknown constant in y_{inner}.

The following graph shows $y_{inner}(x) = e(1 - e^{x/\epsilon})$, $y_{outer}(x) = e^{1-x}$ and the exact solution $y_{exact} = \frac{1}{e^{-1} - e^{-1/\epsilon}}(e^{-x} - e^{-x/\epsilon})$ on the same axes for $\epsilon = 0.1$



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Uniform Approximate Solution

To find a uniformly valid approximate which is valid for the entire domain of interest (from $0 \leq x \leq 1$) instead of a piecewise defined function, we obtain $y_{uniform}(x)$ by adding together the inner and outer solutions and subtracting the common term (the boundary layer constant), so

$$y_{uniform}(x) = y_{inner}(x) + y_{outer} - y_{BLC} \tag{13}$$

Thus

$$y_{uniform}(x) = e(1 - e^{-x/\epsilon}) + e^{1-x} - e = e^{1-x} - e^{1-x/\epsilon}$$

The function which satisfies (1) to leading order, in other words, as $\epsilon \to 0^+$ is

$$y_u(x) = e^{1-x} - e^{1-x/\epsilon}$$

GROUPWORK

Let's show that our uniform solution $y_u(x) = e(e^{-x} - e^{-x/\epsilon})$ satisfies the BVP and the ODE.

$\epsilon \dfrac{d^2 y_u}{dx^2} + (1+\epsilon)\dfrac{dy_u}{dx} + y = 0, \qquad y_u(0) = 0, \quad y_u(1) = 1.$

QUESTION What is the size of the error in the uniform solution and what happens to the error as $\epsilon \to 0^+$?

Theorem 3.12 on page 187 of Logan on provides a formula for obtaining formulas for the inner and outer solution of a standard second order boundary value problem with boundary layer at $x = 0$.

THEOREM

Given

$$\epsilon y'' + p(x)y' + q(x)y = 0, \quad 0 < x < 1, \quad y(0) = a, \quad y(1) = b, \quad 0 < \epsilon \ll 1$$

where $p(x)$ and $q(x)$ are continuous functions on $0 \le x \le 1$ and $p(x) > 0$ for $0 \le x \le 1$. hen there exists a boundary layer at $x = 0$ with inner and outer approximations given by

$$y_{inner}(x) = C_1 + (a - C_1)e^{-p(0)x/\epsilon} \tag{14}$$

$$y_{outer}(x) = b \exp\left(\int_x^1 \frac{q(s)}{p(s)} ds\right) \tag{15}$$

$$C_1 = b \exp\left(\int_0^1 \frac{q(s)}{p(s)} ds\right) \tag{16}$$

GROUPWORK

Use Logan's Theorem 3.12 to obtain an approximate solution to Informal Problem 1 on HW#9.

$$\epsilon y'' + 2y' + y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 1, \quad 0 < \epsilon \ll 1.$$

$\boxed{\text{EXAMPLE}}$

Let's try to use Logan's Theorem 3.12 to generate our inner and outer approximations to the solution to

$$\epsilon y'' + (1+\epsilon)y' + y = 0, \qquad y(0) = 0, \quad y(1) = 1$$

Is there a problem doing this? What will you need to assume?