Senior Colloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am
 `http://sites.oxy.edu/ron/math/400/f20/`

Class 8: Tuesday October 13

TITLE Perturbation Methods on Differential Equations, Part 2: The Poincaré-Lindstedt Method

CURRENT READING Logan, §3.1.3 (pp. 158–159); Witelski, §9.1 (pp. 185–191).

NEXT READING Logan, §3.3 (pp. 179-191); Holmes, §2.5-2.6 (pp. 69-84); Witelski, §7.1-7.3 (pp. 147-158)

SUMMARY
This week we continue looking at regular perturbations in differential equations and stumble upon what can go wrong. We'll be introduced to a method to still produce reasonable perturbation solutions called the Poincaré-Lindstedt method.

Given the following model for a nonlinear spring-mass oscillator

$$m\frac{d^2y}{d\tau^2} = -ky - ay^3, \qquad y(0) = A, \quad \frac{dy}{d\tau}(0) = 0 \tag{1}$$

we can non-dimensionalize it using the scalings

$$u = \frac{y}{A}, \qquad t = \frac{\tau}{\sqrt{m/k}} \tag{2}$$

to produce

$$\frac{d^2u}{dt^2} = -u - \epsilon u^3, \qquad u(0) = 1, \quad u'(0) = 0 \text{ where } \epsilon = \frac{aA^2}{k} \ll 1 \tag{3}$$

The IVP in (3) is known as Duffing's Equation and has no known exact solution.
If we assume a perturbation series solution of the form

$$u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \dots \tag{4}$$

then we will produce a series of differential equations (with initial conditions) of various orders in epsilon (like we have been doing for awhile)

First, look more closely at Duffing's Equation, like this:

$$\frac{d^2u}{dt^2} + u = -\epsilon u^3, \qquad u(0) = 1, \quad u'(0) = 0$$

QUESTION If $\epsilon \ll 1$ what does your mathematical intuition tell you the solution to Duffing's Equation should look like? Bounded? Unbounded? No idea?

EXAMPLE

Let's show what the initial value problems we get for $u_0(t)$ and $u_1(t)$ are:

The $\mathcal{O}(1)$ equation is

$$\frac{d^2u_0}{dt^2} + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0 \tag{5}$$

The $\mathcal{O}(\epsilon)$ equation is

$$\frac{d^2u_1}{dt^2} + u_1 = -u_0^3, \quad u_1(0) = 0, \quad u_1'(0) = 0 \tag{6}$$

The solution to the leading order IVP, the $\mathcal{O}(1)$ term in (5) is

$$u_0(t) = \cos(t) \tag{7}$$

which means that the $\mathcal{O}(\epsilon)$ equation becomes

$$\frac{d^2u_1}{dt^2} + u_1 = -\cos^3(t), \quad u_1(0) = 0, \quad u_1'(0) = 0$$

But using the common trigonometric identity $\cos(3t) = 4\cos^3(t) - 3\cos(t)$ the first-order equation (6) becomes

$$\frac{d^2u_1}{dt^2} + u_1 = -\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t), \quad u_1(0) = 0, \quad u_1'(0) = 0 \tag{8}$$

which can be solved using the Method of Undetermined Coefficients (assume a solution of the form $A\cos(t) + B\sin(t) + C\cos(3t) + Dt\cos(t) + Et\sin(t)$ which produces the following solution (after applying the initial conditions)

$$u_1(t) = \frac{1}{32}\cos(3t) - \frac{1}{32}\cos(t) - \frac{3}{8}t\sin(t) \tag{9}$$

EXAMPLE

Let's confirm the result in (9) using M.O.U.C that $u_1(t) = \frac{1}{32}\cos(3t) - \frac{1}{32}\cos(t) - \frac{3}{8}t\sin(t)$ are the solutions to (8).

Exercise

Confirm that the given functions in (7) and (9) are indeed the solution(s) to the IVPs in (5) and (6), respectively.

Consider a graph of $u_0(t)$ (in red) and $u_0(t) + \epsilon u_1(t)$ (in blue) plotted versus time for a typical value of $\epsilon = 0.1$ on the interval $0 \le t \le \dfrac{1}{\epsilon^2}$. What do you notice?

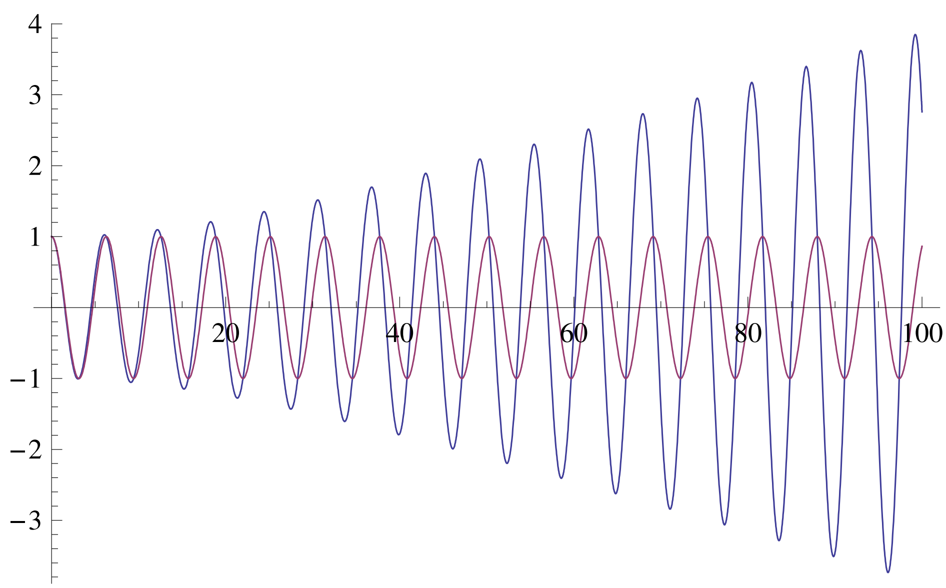

Figure 1: Graph of u_0 in blue and $u_0 + \epsilon u_1$ over time $0 \le t \le 1/\epsilon^2$

Therefore $\epsilon u_1(t)$ is NOT much less than $u_0(t)$ for all time. (**If the solutions were informly valid what would you expect the graphs to look like?**) Can you explain what happens as t gets larger and larger? Is it possible to estimate the value of t where "trouble" begins?

QUESTION **Explain the significance of the Figure**

The Poincaré-Lindstedt Method

In this technique the perturbation series is chosen to be

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \dots \tag{10}$$

where $\tau = \omega t$ and

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots \tag{11}$$

We can choose $\omega_0 = 1$ since it is the frequency of the solution given in (7) to the leading-order problem in Equation (6).

Using these new scalings given in (10) and (11) we can transform Duffing's Equation (3)

$$u'' = -u - \epsilon u^3, \qquad u(0) = 1, \quad u'(0) = 0)$$

into

$$\omega^2 \frac{d^2u}{d\tau^2} = -u - \epsilon u^3, \qquad u(0) = 1 \quad u'(0) = 0 \tag{12}$$

EXAMPLE

First let's show how we get to (12) from (3) using $\tau = \omega t$

We can can obtain the ordered equations from

$$\omega^2 \frac{d^2 u}{d\tau^2} = -u - \epsilon u^3, \qquad u(0) = 1 \quad u'(0) = 0$$

remebering $\omega = \omega_0 + \epsilon\omega_1 + \ldots$ and $u = u_0(\tau) + \epsilon u_1(\tau) + \ldots$ and $\tau = \omega t$

The $\mathcal{O}(1)$ equations are

$$\omega_0^2 \frac{d^2 u_0}{d\tau^2} + u_0 = 0, \quad u_0(0) = 1, \omega_0 \quad u_0'(0) = 0 \tag{13}$$

The $\mathcal{O}(\epsilon)$ equations are

$$-2\omega_1\omega_0 u_0'' + \omega_0^2 \frac{d^2 u_1}{d\tau^2} + u_1 = -u_0^3, \quad u_1(0) = 0, \quad \omega_0 u_1'(0) + \omega_1 u_0'(0) = 0 \tag{14}$$

The solution to (13), $\dfrac{d^2u_0}{d\tau^2} + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0$, is similar to the solution from (5) which turns out to be

$$u_0(\tau) = \cos(\tau) \tag{15}$$

which leads to the $\mathcal{O}(\epsilon)$ equation in (14) becoming

$$\frac{d^2u_1}{d\tau^2} + u_1 = \left(2\omega_1 - \frac{3}{4}\right)\cos(\tau) - \frac{1}{4}\cos(3\tau), \quad u_1(0) = u_1'(0) = 0 \tag{16}$$

NOTE that Equation (16) is solved using the same techniques for Equation (8), with the extra term $2\omega_1\cos(\tau)$ coming from $-2\omega_1 u_0''$.
In order to eliminate the $\cos(\tau)$ term on the right-hand side of (16) we can let $2\omega_1 - \frac{3}{4} = 0$ so $\omega_1 = \dfrac{3}{8}$ which produces

$$\frac{d^2u_1}{d\tau^2} + u_1 = -\frac{1}{4}\cos(3\tau)$$

We can again use the Method of Undetermined Coefficients and the initial conditions to show that the solution to the above equation is

$$u_1(\tau) = \frac{1}{32}[\cos(3\tau) - \cos(\tau)] \text{ where } \tau = t + \frac{3}{8}\epsilon t + \dots \tag{17}$$

A first-order, uniformly-valid perturbation solution of Duffing's Equation (3) is $u_0(\tau) + \epsilon u_1(\tau)$,

$$u(\tau) = \cos(\tau) + \frac{1}{32}\epsilon[\cos(3\tau) - \cos(\tau)] \text{ where } \tau = t + \frac{3}{8}\epsilon t + \dots \tag{18}$$

A graph of (18), a 2-term perturbation solution of Duffing's Equation versus time for a typical value of $\epsilon = 0.1$ on the interval $0 \leq t \leq \dfrac{1}{\epsilon^2}$ is shown below. NOW what do you notice?

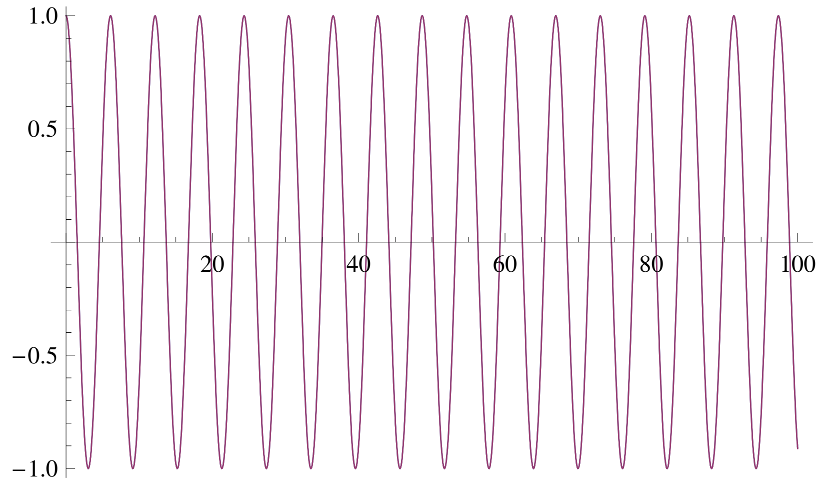

Figure 2: Graph of 2-term perturbation solution to Duffings Equation using scaled time

Here's a graph of the difference between $u(\tau)$ and $u_0(\tau)$ which equals $\epsilon u_1(\tau)$ on the same interval $0 \leq t \leq \dfrac{1}{\epsilon^2}$

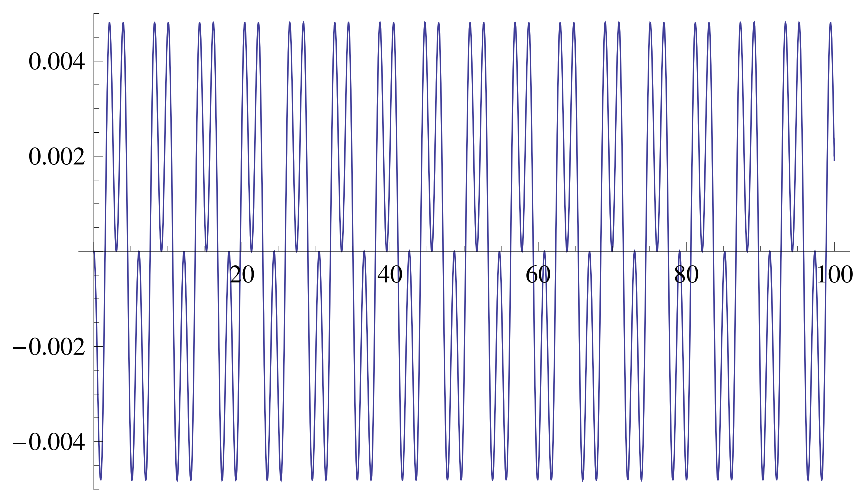

Figure 3: Graph of $\epsilon u_1(t)$ versus time

QUESTION **EXPLAIN the significance of the above Figures**