Senior Colloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am
 `http://sites.oxy.edu/ron/math/400/f20/`

Class 7: **Tuesday October 6**

TITLE (Regular) Perturbation Methods on Differential Equations

CURRENT READING Logan, §3.2.2 (pp. 173–174) & §3.1.1 (pp. 153–155); Holmes §2.2 (pp. 60-63); Witelski, §6.4 (pp. 138–139).

NEXT READING Logan, §3.1.3 (pp. 158–159); Witelski, §9.1 (pp. 185–191).

SUMMARY
Previously we looked at what happens to their solutions when small perturbations are introduced into algebraic equations. Now we are going to begin looking at what happens to solutions of differential equations when they have small-valued parameters that are perturbed. We also need to think about asymptotic expansions when the s

Perturbations of ODEs

Consider the differential equation for a body of mass m moving in a straight line with initial velocity V_0 subject to a resistive force (*cough* bomb drop *cough*) results in the following IVP

$$m\frac{dv}{d\tau} = -av + bv^2, \qquad v(0) = V_0 \tag{1}$$

We can non-dimensionalize the IVP using

$$y = \frac{v}{V_0}, \qquad t = \frac{\tau}{m/a} \tag{2}$$

to produce

$$\frac{dy}{dt} = -y + \epsilon y^2, \qquad y(0) = 1 \text{ where } \epsilon = \frac{bV_0}{a} \ll 1 \tag{3}$$

The ODE in (3) is in a class of differential equations of the form $y' + p(x)y = q(x)y^n$ called Bernoulli Equations (named after the famous Bernoulli Brothers Jacob and Johann who were instrumental in the development of Fluid Mechanics) where "the trick" is to change variables through the substitution $w = \dfrac{1}{y}$ to produce a linear equation

$$\frac{dw}{dt} = w - \epsilon, \quad w(0) = 1$$

which can be solved to produce the exact solution below

$$y(t) = \frac{e^{-t}}{1 + \epsilon(e^{-t} - 1)} \tag{4}$$

Exercise

Confirm that the given solution in (4) is indeed the solution to the IVP in (3)

EXAMPLE

Let's also show that we can do a Taylor Expansion of the exact solution in (4) to produce an approximation which looks like

$$y_{exact}(t) = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \dots \tag{5}$$

Perturbation Series Solution

Let's assume that the solution to the IVP in (3), $y' = -y + \epsilon y^2, \quad y(0) = 1$ has a perturbation series solution like we have been assuming previoulsy for algebraic equations.

$$y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots \tag{6}$$

Then we will produce a series of differential equations (with initial conditions) of various orders in epsilon...
We can show that the systems we get are:
The $\mathcal{O}(1)$ equation is

$$y_0' = -y_0, \quad y(0) = 1 \tag{7}$$

The $\mathcal{O}(\epsilon)$ equation is

$$y_1' = -y_1 + y_0^2, \quad y_1(0) = 0 \tag{8}$$

The $\mathcal{O}(\epsilon^2)$ equation is

$$y_2' = -y_2 + 2y_0y_1, \quad y_2(0) = 0 \tag{9}$$

The individual IVPs for $y_0(t)$, $y_1(t)$ and $y_2(t)$ is

$$\begin{aligned} y_0' &= -y_0, \qquad y(0) = 1 \\ y_1' &= -y_1 + y_0^2, \qquad y_1(0) = 0 \\ y_2' &= -y_2 + 2y_0y_1, \qquad y_2(0) = 0; \end{aligned}$$

Show that our 3-term approximation $y_{approx}(t)$ to the solution of (3), $y' = -y + \epsilon y^2, \quad y(0) = 1.$

$$y_{approx}(t) = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \dots \tag{10}$$

Look familiar?

The point here is that $y_{exact} - y_{approx} = m_1(t)\epsilon^3 + m_2(t)\epsilon^4 + \dots$ for $t > 0$ where $m_1, m_2, \dots$ are bounded functions so that as $\epsilon \to 0$ this difference will go to zero for all positive values of t.

Validity of Asymptotic Expansions

DEFINITION: **asymptotic sequence**

A sequence of gauge functions $\{g_n(t,\epsilon)\}$ is called an **asymptotic sequence** as $\epsilon \to 0$ if for all t in some interval $\mathcal{I}$, $g_{n+1+}(t,\epsilon) = o(g_n(t,\epsilon))$ as $\epsilon \to 0$ for $n = 0, 1, 2, \ldots$.

TRANSLATION Every element in an asymptotic sequence goes to zero faster than the term that precedes it. For example, in the standard asymptotic sequence we have been looking at $y \approx y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots$.

DEFINITION: **asymptotic expansion**

Given a function $y(t,\epsilon)$ and an asymptotic sequence $\{g_n(t,\epsilon)\}$ as $\epsilon \to 0$, the series $\sum_{n=0}^{\infty} a_n g_n(t,\epsilon)$ is said to be an **asymptotic expansion** of $y(t,\epsilon)$ as $\epsilon \to 0$ if for every N

$$y(t,\epsilon) - \sum_{n=0}^{N} a_n g_n(t,\epsilon) = o(g_N(t,\epsilon)) \text{ as } \epsilon \to 0$$

TRANSLATION The difference between the partial sum and the function being approximated, i.e. the remainder, always goes to zero faster than the last term in the partial sum.
Note that in most cases, an asymptotic expansion has the form of a product of functions of t and ϵ, so that $g_n(t,\epsilon) = g_n(t)\phi_n(\epsilon)$. In that case, we often write

$$y \sim \sum_{n=0}^{\infty} a_n g_n(t,\epsilon) \text{ or } y \sim \sum_{n=0}^{\infty} g_n(t)\phi_n(\epsilon)$$

and say y is being represented/approximated by the asymptotic expansion on the right side of the $\sim$ symbol.

DEFINITION: **uniformly valid**

A function $y_a(t,\epsilon)$ is said to be a **uniformly valid asymptotic approximation** to $y(t,\epsilon)$ on an interval $\mathcal{I}$ as $\epsilon \to 0$ if the error $E(t,\epsilon) \equiv y(t,\epsilon) - y_a(t,\epsilon)$ converges to zero **uniformly** for every $t \in \mathcal{I}$

TRANSLATION For a function $f(t,\epsilon)$ to converge uniformly to a limit L (say, zero) as $\epsilon \to 0$ on $t \in \mathcal{I}$ means that you have to be able to show that for **any** t in the interval $\mathcal{I}$ you can show that $f(t,\epsilon) \to L$ as $\epsilon \to 0$. If the $f(t,\epsilon) \to L$ for fixed values of t (say, t_0) then you say that f has *pointwise convergence* to its limit L.

EXAMPLE

Show that the expression $y_a(t,\epsilon) = 1 - \epsilon t$ is **not** a uniformly valid asymptotic expansion of the function $y(t,\epsilon) = e^{-\epsilon t}$ as $\epsilon \to 0$ on the interval $t > 0$