Senior **C**olloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am
 `http://sites.oxy.edu/ron/math/400/f20/`

Class 6: Tuesday September 29

TITLE Singular Perturbation Methods on Algebraic Equations

CURRENT READING Logan, §3.2.1 (pp. 170-173); Holmes, §2.1-2.2 (pp. 49–60);

NEXT READING Logan, §3.2.2 (pp. 173–174) & §3.1.1 (pp. 153–157); Holmes §2.2 (pp. 60-63); Witelski, §6.4 (pp. 138–139).

SUMMARY
This week we will be introduced into the wonderful world of perturbation methods. We shall begin by looking at what can happen to simple algebraic equations when a small term is included.

Singular Perturbation Of Algebraic Equations

In the case where the unperturbed ($\epsilon = 0$) problem has a quantitatively different nature than the perturbed problem we are usually dealing with a singular perturbation problem.

EXAMPLE

Consider Example 3.7 on page 170 of Logan

$$\epsilon x^2 + 2x + 1 = 0, \qquad 0 \leq \epsilon \ll 1 \tag{1}$$

Notice what happens when you consider the unperturbed version of (1) by solving the problem with $\epsilon = 0$ and considering $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$

The $\mathcal{O}(\epsilon^0)$ equation is

$$2x_0 + 1 = 0 \tag{2}$$

so that $x_0 = -\frac{1}{2}$ because the leading order equation is **linear** instead of quadratic! It will produce fewer solutions than the original perturbed problem in (1). That is a tell-tale sign of a regular perturbation problem.

Note that we can still obtain the higher-order equations to obtain values for x_1 and x_2...
The $\mathcal{O}(\epsilon)$ equation is

$$x_0^2 + 2x_1 - 1 = 0 \tag{3}$$

The $\mathcal{O}(\epsilon^2)$ equation is

$$2x_1 x_0 + 2x_2 = 0 \tag{4}$$

which leads to only one solution to the given perturbed quadratic equation in (1)

$$x = -\frac{1}{2} - \frac{1}{8}\epsilon - \frac{1}{16}\epsilon^2 + \dots \tag{5}$$

The Hard Way: Dominant Balancing

To find the other solution we must "re-balance" terms by making a new estimate (6) for the perturbation series and plug it into (1) instead

$$x = x_0 + \epsilon^{\nu} x_1 + \epsilon^{2\nu} x_2 + \dots \tag{6}$$

In general we can really just keep the first two terms $x = x_0 + \epsilon^{\nu} x_1$ in (6) and show that this gives us the same $\mathcal{O}(1)$ equation as given before in (2) but the following equation for the next order term produces

$$x_0^2\epsilon + x_1^2\epsilon^{2\nu+1} + 2x_0x_1\epsilon^{\nu+1} + 2x_1\epsilon^{\nu} = 0 \tag{7}$$

Write a number under each of the four terms in equation (7). What follows now is that we have to look at all the possibilities of balancing any two of these terms with each other and solving for ν. The correct balancing will produce a well-ordered perturbation series. There are $\binom{4}{2}$ possibilities (i.e. six).

$$\underbrace{x_0^2\epsilon}_{①} + \underbrace{x_1^2\epsilon^{2\nu+1}}_{②} + \underbrace{2x_0x_1\epsilon^{\nu+1}}_{③} + \underbrace{2x_1\epsilon^{\nu}}_{④} = 0$$

GROUPWORK

Go through all the possible balancings and solve for ν and explain why this balancing is consistent or inconsistent in each case.

CASE I: ① ≈ ②

CASE II: ① ≈ ③

CASE III: ① ≈ ④

CASE IV: ② ≈ ③

CASE V: ② ≈ ④

CASE VI: ③ ≈ ④

The correct balancing is TERM (2) and TERM (4) which leads to $\nu = -1$. This results in the following equations

$$
\begin{aligned}
2x_0 + 1 + 2x_0 x_1 &= 0 \qquad \mathcal{O}(1) &(8)\\
2x_1 + x_1^2 &= 0 \qquad \mathcal{O}\left(\frac{1}{\epsilon}\right) &(9)
\end{aligned}
$$

(NOTE: the expression in blue comes from the original zeroth-order expansion of 1 to produce equation (2) and the expression in red is now also order 1 because of our choice for $\nu = -1$ produces an extra zeroth order term.

The system of equations in (8) and (9) has solutions $x_0 = -1/2, x_1 = 0$ and $x_0 = 1/2, x_1 = -2$. This produces a 2-term approximation the second root of equation (1), i.e. $\epsilon x^2 + 2x + 1 = 0$ to be

$$x = -\frac{2}{\epsilon} + \frac{1}{2} + \dots \qquad (10)$$

Recall, the first root was given in (5) and is $x = -\frac{1}{2} - \frac{1}{8}\epsilon - \frac{1}{16}\epsilon^2 + \dots$

Exercise

Solve the system of equations in (8) and (9).

An Easier Way: Use Balancing To Revert To A Regular Perturbation Problem
We could have just looked at the three terms in $\epsilon x^2 + 2x + 1 = 0$ and looked at the two possible balancings in which you keep two of the terms in the equation and ignore the third because it is very much smaller than the other two when $\epsilon \ll 1$.

$$\epsilon x^2 + 2x + 1 = 0$$
$$\textcircled{1} \quad \textcircled{2} \quad \textcircled{3}$$

The first balancing choice is $\textcircled{1}$ with $\textcircled{3}$, i.e. $\epsilon x^2 \approx 1$ which produces $x \approx \mathcal{O}(1/\sqrt{\epsilon})$ which means that the first and third terms will be the same size (and $\mathcal{O}(1)$ while the middle term will be $\mathcal{O}(1/\sqrt{\epsilon})$ which is much larger (since $\epsilon \ll 1$) so therefore the second term **CAN NOT** be ignored. Thus this balancing choice is rejected.

The second balancing choice is $\textcircled{1}$ with $\textcircled{2}$, or $\epsilon x^2 \approx 2x$ and one finds that $x = \mathcal{O}\left(\frac{1}{\epsilon}\right)$. This is the only one that "makes sense." In other words, the first and second terms will be $\mathcal{O}(1/\epsilon)$ and the third term will be $\mathcal{O}(1)$ which is much *smaller* when ϵ is very small so this term can be ignored. This means that one could re-scale x so that $y = \dfrac{x}{1/\epsilon}$ and re-substitute $x = y/\epsilon$ into (1) to produce

$$y^2 + 2y + \epsilon = 0 \tag{11}$$

which is a regular perturbation problem that can be solved using a perturbation series of the form $y = y_0 + \epsilon y_1 + y_2\epsilon^2 + \ldots$.

Exercise

Use this scaling (remembering that $y = \epsilon x$) and show that one obtains the same 2-term approximations for the roots of $\epsilon x^2 + 2x + 1 = 0$ as given before. In fact, go further and obtain a 3-term expansion for this root.

The Easiest Way: Quadratic Formula Plus Taylor Expansion

Find a three-term expansion for the roots of $\epsilon x^2 + 2x + 1 = 0$ by using the quadratic formula and the Taylor Expansion for $\sqrt{1+\epsilon} \approx 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)$

TUTORIAL PROBLEM

Show that $f(\epsilon) = \ln(1+\sqrt{\epsilon}) \approx \sqrt{\epsilon} - \frac{1}{2}\epsilon + \frac{1}{3}\epsilon^{3/2} + \mathcal{O}(\epsilon^2)$

Visualization of Perturbation of Algebraic Equations

If one considers solutions of equations as the intersection of curves then the geometrical interpretation of what we have been doing can be visualized at this website:

`https://www.desmos.com/calculator/7pafjh8ue5`

Consider the solution(s) of $\epsilon x^2 + 2x + 1 = 0$ as the intersection of the graph $y_1 = -2x - 1$ and $y_2 = \epsilon x^2$ graphed below with $\epsilon = .25$

RECALL The roots of the equation are $x = -\frac{1}{2} - \frac{1}{8}\epsilon - \frac{1}{16}\epsilon^2 + \ldots$ and $x = -\frac{2}{\epsilon} + \frac{1}{2} + \frac{1}{8}\epsilon + \ldots$

QUESTION Where should the two graphs intersect when $\epsilon = 0.25$? How about when $\epsilon = 0.1$?