Senior **C**olloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am
 `http://sites.oxy.edu/ron/math/400/f20/`

Class 5: Tuesday September 22

TITLE Introduction to Perturbation Methods and Asymptotic Expansions

CURRENT READING Logan, §3.1.1 (pp. 149-152); Logan, §3.1.4 (pp. 160-165); Witelski, §6.1-6.3 (pp. 127-137); Holmes, §2.1-2.2 (pp. 49–60);

NEXT READING Logan, §3.2.1 (pp. 170-173); Holmes, §2.1-2.2 (pp. 49–60);

SUMMARY

This week we will be introduced into the wonderful world of perturbation methods. After a review of "big Oh" and "little Oh" and asymptotic expansions we shall begin by looking at applying these ideas to some polynomial equations when a small term is included.

Order of Convergence of Functions

DEFINITION: **"big Oh" or $\mathcal{O}$**

If we know that $\lim_{h\to 0} F(h) = L$ and $\lim_{h\to 0} G(h) = 0$ and if a positive constant K exists with

$$|F(h) - L| \leq KG(h), \qquad \text{for sufficiently small } h$$

then we write $F(h) = L + \mathcal{O}(G(h))$ and say "$F(h)$ is big Oh of $G(h)$ as h goes to zero."

This can also be computed using the idea that $F(h) = L + \mathcal{O}(G(h))$ if and only if

$$\lim_{h\to 0} \frac{|F(h) - L|}{|G(h)|} = K$$

where K is some positive, finite constant.

Basically this means in a "race" between $F(h) \to L$ and $G(h) \to 0$ there would be a tie–they approach their respective limits **at about the same rate**.

DEFINITION: **"little Oh" or o**

$F(h) = L + o(G(h))$ if and only if

$$\lim_{h\to 0} \frac{|F(h) - L|}{|G(h)|} = 0$$

Basically this means that in a "race" between $F(h) \to L$ and $G(h) \to 0$, F will always win it. In other words, $F(h)$ approaches its limit of L at a **faster** rate than $G(h)$ approaches its limit of 0.

The function $G(h)$ is known as a **gauge function** and typically has a "nice" form such as a single-term polynomial of the form h^p where p is a whole number.

This means that if $F(h) = L + o(G(h))$ as $h \to 0$ then we say that $|F(h) - L| \ll G(h)$ as $h \to 0$

EXAMPLE

Show that the expression $\cos(h) + \frac{h^2}{2}$ is $1 + \mathcal{O}(h^4)$ as $h \to 0$.

EXERCISE

Show that $f(n) = e^{-n} = o(n^{-p})$ for all $p > 0$ as $n \to \infty$.

Show $e^{\epsilon} - 1 = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$.

Question **Discuss the usefulness of the terms $f(\epsilon) = o(1)$ and $g(\epsilon) = \mathcal{O}(1)$ as $\epsilon \to 0$. What can you say about f and g?**

Taylor Expansions
Recall that if you have a function $f(x)$ which is infinitely-differentiable, you can write down the behavior of $f(x)$ near $x = 0$ with a MacLaurin Series (also called a Taylor Series):

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$$

If we are only interested in the behavior of the function for small values of x near 0, i.e. for $|h| << 1$ then we can write the expression as

$$f(0+h) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{h^k}{k!}$$

Interestingly, we can write an exact expression for the truncated form of this infinite series as

$$f(0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2} + \mathcal{O}(h^3)$$

Yes, this last term is the same "big oh" that we have been discussing in regards to order of convergence of a function to its limit.

Table 1.1: Some commonly used Taylor Expansions around $x = 0$

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{3!}x^3 f'''(0) + \cdots$$

$$(a+x)^\gamma = a^\gamma + \gamma x a^{\gamma-1} + \frac{1}{2}\gamma(\gamma-1)x^2 a^{\gamma-2} + \frac{1}{3!}\gamma(\gamma-1)(\gamma-2)x^3 a^{\gamma-3} + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots$$

$$\sin(a+x) = \sin(a) + x\cos(a) - \frac{1}{2}x^2\sin(a) - \frac{1}{6}x^3\cos(a) + \cdots$$

$$\cos(a+x) = \cos(a) - x\sin(a) - \frac{1}{2}x^2\cos(a) + \frac{1}{6}x^3\sin(a) + \cdots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

$$\ln(a+x) = \ln(a) + \ln(1+x/a) = \ln(a) + \frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 + \cdots$$

Multiple Techniques For Computing Order of Convergence of a Function

Let's use multiple methods to show that $f(h) = A + \mathcal{O}(h^{\alpha}) = B + o(h^{\beta})$ for $f(h) = \dfrac{1 + h - e^{h}}{h^{2}}$

1. **Limit Method**

2. **Bounding/Inequality Method**

3. **Truncated Taylor/Maclaurin Expansion**

(For each of the methods above, write a short note to yourself explaining the method.)

Regular Perturbation Of Algebraic Equations

Consider the equation

$$x^2 + 2\epsilon x - 3 = 0 \tag{1}$$

Considering a perturbation series of the form $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ and writing down equations for each order, i.e. powers of ϵ, produces

The $\mathcal{O}(1)$ equation is

$$x_0^2 - 3 = 0 \tag{2}$$

The $\mathcal{O}(\epsilon)$ equation is

$$2x_0(x_1 + 1) = 0 \tag{3}$$

The $\mathcal{O}(\epsilon^2)$ equation is

$$x_1^2 + 2x_0 x_2 + 2x_1 = 0 \tag{4}$$

Solving (2), (3) and (4) in order produces the solutions $x_0 = \pm\sqrt{3}$, $x_1 = -1$ and $x_2 = \pm\dfrac{1}{2\sqrt{3}}$ that corresponds to

$$x = \sqrt{3} - \epsilon + \frac{1}{2\sqrt{3}}\epsilon^2 + \ldots \text{ and } x = -\sqrt{3} - \epsilon - \frac{1}{2\sqrt{3}}\epsilon^2 + \ldots$$

EXAMPLE

Let's show this result

Exercise

Show that the solutions to $x^2 + 2\epsilon x - 3 = 0$ are

$$x = \sqrt{3} - \epsilon + \frac{1}{2\sqrt{3}}\epsilon^2 + \dots \text{ and } x = -\sqrt{3} - \epsilon - \frac{1}{2\sqrt{3}}\epsilon^2 + \dots$$

by using the quadratic formula on the equation (1) and then applying a Taylor Expansion...