Senior Colloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am

 `http://sites.oxy.edu/ron/math/400/f20/`

Class 4: Tuesday September 15

TITLE	***Scaling or Non-Dimensionalization, Part 2***
CURRENT READING	Logan, §1.2 (pp. 30-40); Holmes, §1.5 (pp. 27-34)
NEXT READING	Logan, §3.1.1 (pp. 149-152); Holmes, §2.1-2.2 (pp. 49–60); Witelski, §6.1-6.3 (pp. 127-137)

SUMMARY
This week we will continue looking at scaling problems by examining the rather complicated (but famous) projectile problem

The Projectile Problem

Let us analyze the motion of a projectile thrust into the atmosphere from the surface of the earth vertically. It turns out the governing equation is Newton's second law of motion, $F = ma$ which looks like

$$m\frac{d^2h}{dt^2} = -G\frac{Mm}{(h+R)^2} \tag{1}$$

Where R is the radius of the earth, M is its mass, m is the mass of the projectile, V its velocity and h its height. On the earth's surface, i.e. at $h = 0$ the weight of the projectile is equal to the gravitational force, so $mg = \dfrac{GMm}{R^2}$ or $g = \dfrac{GM}{R^2}$.
Thus we can use $GM = gR^2$ in Equation (1) to produce

$$\frac{d^2h}{dt^2} = -\frac{R^2g}{(h+R)^2} \tag{2}$$

with initial conditions

$$h(0) = 0 \quad h'(0) = V \tag{3}$$

These ODE in (2) plus the equation in (3) represents an initial value problem which is the mathematical model for this projectile problem.

The book (Logan 36-39) does a non-dimensional analysis of the variables involved

$[t]$= time T

$[h]$= length L

$[R]$ = length L

$[V]$= velocity LT^-1

$[g]$= acceleration LT^{-2}

assuming there exists a physical law $f(t, h, R, L, V, g) = 0$ relating these variables, It turns out that there are three dimensionless quantities formed from t, h, R, L, V, and g.

$$\pi_1 = \frac{h}{R} \qquad \pi_2 = \frac{t}{R/V} \qquad \pi_3 = \frac{V}{\sqrt{gR}}$$

which implies that $F(\pi_1, \pi_2, \pi_3) = 0$ or $\pi_1 = F_1(\pi_2, \pi_3)$ or

$$\frac{h}{R} = F_1\left(\frac{t}{R/V}, \frac{V}{\sqrt{gR}}\right)$$

Scaling The Projectile Problem
We need to choose a characterisic time t_c and a characteristic length h_c where

$$\tilde{t} = \frac{t}{t_c} \text{ and } \tilde{h} = \frac{h}{h_c}$$

Interestingly, there are three choices of pairs for t_c and h_c which are physically meaningful. Let's look at the choices and see how they change the IVP rpresenting the model given in (2) and (3).

CHOICE 1

$$\tilde{t} = \frac{t}{R/V}, \tilde{h} = \frac{h}{R} \tag{4}$$

CHOICE 2

$$\tilde{t} = \frac{t}{\sqrt{R/g}}, \tilde{h} = \frac{h}{R} \tag{5}$$

CHOICE 3

$$\tilde{t} = \frac{t}{V/g}, \tilde{h} = \frac{h}{V^2/g} \tag{6}$$

Question **How many other choices for h_c and t_c are there? What makes our specific three choices above "physically meaningful"?**?

EXAMPLE

Let's use each of these choices to demonstrate what happens to the IVP for the projectile model.

CHOICE 1 $\tilde{t} = \dfrac{t}{R/V}, \tilde{h} = \dfrac{h}{R}$

The IVP becomes

$$\epsilon \frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{h})^2}, \qquad \tilde{h}(0) = 0, \quad \frac{d\tilde{h}}{d\tilde{t}} = 1 \tag{7}$$

Question **What collection of variables corresponds to ϵ in Equation 7**?

CHOICE 2 $\tilde{t} = \dfrac{t}{\sqrt{R/g}}, \tilde{h} = \dfrac{h}{R}$

The IVP becomes

$$\frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{h})^2}, \qquad \tilde{h}(0) = 0, \quad \frac{d\tilde{h}}{d\tilde{t}} = \sqrt{\epsilon} \tag{8}$$

Question **What collection of variables corresponds to ϵ in Equation 8**?

CHOICE 3 $\tilde{t} = \dfrac{t}{V/g}, \tilde{h} = \dfrac{h}{V^2/g}$

The IVP becomes

$$\frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\epsilon\tilde{h})^2}, \qquad \tilde{h}(0) = 0, \quad \frac{d\tilde{h}}{d\tilde{t}} = 1 \tag{9}$$

Question **What collection of variables corresponds to ϵ in Equation 9**?

GROUPWORK

Discuss what happens to each of the problems given in Equation 7, Equation 8 and Equation 9 when $\epsilon << 1$, i.e. as $\epsilon \to 0$