Senior Colloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am

 `http://sites.oxy.edu/ron/math/400/f20/`

Class 3: **Tuesday September 8**

TITLE Introduction to Scaling and Nondimensionalization

CURRENT READING Logan, §1.2 (pp. 30-40); Holmes, §1.5 (pp. 27-34)

NEXT READING Logan, §1.2 (pp. 30-40); Holmes, §1.5 (pp. 27-34)

SUMMARY

This week we will be introduced to the concept of characteristic scales and the importance of scaling in real-world problems.

DEFINITION: **Scaling**

The process of selecting new, usually dimensionless variables and re-formulating the problem in terms of those new variables (Logan 30).

Sometimes the process is called **nondimensionalization**. The Buckingham Pi Theorem assures us that we can always find a non-dimensional (scaled) version of a given problem.

Scaling is most commonly used on the time variable. Many real world processes occur over various time scales, such as in a chemical reaction where one might have small changes in concentration over a relatively long period of time, and then suddenly a tipping point is reached and a very fast change in concentration happens very very quickly.

Generally one accomplishes a scaling in time by selecting a characteristic time value t_c and making a dimensionless version of time $\bar{t}$ by using the following equation $\bar{t} = \frac{t}{t_c}$. If the characteristic time scale is chosen correctly then $\bar{t}$ should be on the order of unity, not "too small" or "too big"–sorta like "Goldilocks"! In biological problems one can have different length-scales which vary incredibly widely "as much as 10^{15} orders of magnitude" (Logan 30).

NOTATION Sometimes one uses $\tilde{x}$ to represent the unscaled (original) variable and x becomes the scaled variable. Logan generally uses $\bar{x}$ to represent the scaled variable and x to represent the original (unscaled) one.

Population Models

The **Malthus Population model** states that the growth rate of a population is proportional to its current population, i.e.

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$$

This results in exponential growth! It also means that the per capita growth rate (the rate divided by the total population) is a constant value k.

The **Verhulst Population model** (sometimes known as the Logistic Model) is a modification of the Malthusian model which changes the per capita growth rate from a constant to a rate that decreases linearly as population increases to a maximum value, called the carrying capacity M, or

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

EXAMPLE

(Example 1.11, Logan, p. 31)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \qquad P(0) = P_0 \tag{1}$$

Let's scale this problem by producing a non-dimensional version of the problem. We need to select dimensionless versions of the dependent (P) and independent variables (t). We'll form a characteristic value P_c and t_c from the given constants in the problem: P_0, M and k.
Use $P_c = M$ and $t_c = \dfrac{1}{k}$ so that $\bar{P} = \dfrac{P}{m}$ and $\bar{t} = \dfrac{t}{t_c} = \dfrac{t}{\frac{1}{k}} = kt$.

This produces the nondimensional version of (1) which looks like

$$\frac{d\bar{P}}{d\bar{t}} = \bar{P}(1-\bar{P}), \qquad \bar{P}(0) = \alpha \tag{2}$$

where $\alpha = \dfrac{P_0}{M}$ which is a dimensionless constant.

Note that the solution to Equation (2) is an unknown function $\bar{P}(\bar{t})$ while the solution to the original problem given in Equation (1) is an unknown function $P(t)$. The relationship between the two is $P(t) = P_c\bar{P}(\bar{t})$ and $t = t_c\bar{t}$.

Question **What's the physical interpretation of the constant α? What does it signify?**

Exercise

Show that if you select a different scaling (i.e. $P_c = P_0$ and $t_c = \frac{1}{k}$) one obtains the following dimensionless equation:

$$\frac{d\bar{P}}{d\bar{t}} = \bar{P}(1-\beta\bar{P}), \qquad \bar{P}(0) = 1 \tag{3}$$

where $\beta = \dfrac{P_0}{M}$ is a dimensionless constant.

EXAMPLE

The solution to the dimensional equation in Equation (1) is given by

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kt}}$$

Let's confirm that this function does indeed solve the initial value problem (IVP) given in Equation (1).

GROUPWORK

Use the previous EXAMPLE to write down the solution of the dimensionless problems given in Equation (2) and in Equation (3).

Question **In each case what is the value $\lim_{t\to\infty} P(t)$ and what is its significance?**