*Class 2*: **Tuesday September 1**

**TITLE** The Buckingham Pi Theorem (Dimensional Analysis, Continued)
**CURRENT READING** Logan, §1.1, (pp. 8-29);
**NEXT READING** Logan, §1.2 (pp. 30-40); Holmes, §1.5 (pp. 27-34)

**SUMMARY**
This week we will do some more complicated dimensional analysis and look at the Buckingham Pi Theorem more formally.

**RECALL**
The informal statement of the Pi Theorem: "Given a physical law that gives a relation among a certain number of *dimensioned* quantities, then there is an equivalent law that can be expressed as a relation among certain **dimensionless** quantities" (Logan 5).

**The Buckingham Pi Theorem**

On page 10 of Logan the Pi Theorem is formally presented:

> Let
> $$f(q_1, q_2, q_3, \dots, q_m) = 0 \tag{1}$$
> be a unit-free physical law that relates the dimensional quantities $q_1, q_2, \dots, q_m$. Let $L_1, L_2, \dots, L_n$ (with $n < m$) be fundamental dimensions with
> $$[q_i] = L_1^{a_{1i}} L_2^{a_{2i}} \dots L_n^{a_{ni}}, \text{ where } i = 1, \dots, m$$
> and let $r = \text{rank}(A)$, where $A$ is an $n \times m$ dimension matrix with entries $a_{ij}$. THEN there exists $m - r$ independent dimensionless quantities $\pi_1, \pi_2, \pi_3, \dots, \pi_{m-r}$ that can be formed from $q_1, q_2, q_3, \dots q_m$ and the physical law (1) is equivalent to another equation
> $$F(\pi_1, \pi_2, \pi_3, \dots, \pi_{m-r}) = 0 \tag{2}$$
> expressed only in terms of **dimensionless** variables.

What does this all mean? Let's look at the theorem closely:
There are really two parts to it

(i) Among the quantities $q_1$, $q_2, \dots, q_m$ there are $m - r$ independent dimensionless variables that can be formed, where $r$ is the rank of the dimension matrix $A$

(ii) If $\pi_1$, $\pi_2$, ..., $\pi_{m-r}$ are the $m - r$ dimensionless variables, then $f(q_1, q_2, q_3, \dots, q_m) = 0$ in Equation (1) is equivalent to a physical law of the form $F(\pi_1, \pi_2, \pi_3, \dots, \pi_{m-r}) = 0$ given above in Equation (2).

Proving Statement (i) is straightforward. Once you set

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2} q_2^{\alpha_3} \ldots q_m^{\alpha_m} \tag{3}$$

We can obtain a homogeneous set of $n$ linear equations in $m$ variables $\alpha_1$, $\alpha_2$, …, $\alpha_m$. We are looking for the set of solutions that span the nullspace of an $n \times m$ matrix with rank $r$, which is $m - r$.
(Note that Logan describes his matrices using $n$ times $m$ instead of the customary $m$ rows and $n$ columns some of you may be used to from Linear Algebra).

Question **What's the nullspace of a matrix $A$ and how do you compute it?**

Proving Statement (ii) is a bit more "subtle" so we will only consider it for a specific example which is about the law $f(x, t, g) = x - \frac{1}{2}gt^2 = 0$ and showing that one can produce a new law $F(\pi_1) = 0$ which is non-dimensionless and equivalent. Let's do that.

**Exercise**
**(Logan, Example 1.10, page 23)**

GROUPWORK

**(Logan, Example 1.7 on page 13)**. Consider a physical law of the form $f(t, r, u, e, k, c) = 0$ where $t$ is time, $e$ is energy, $r$ is radial distance from the heat source, $c$ is the heat capacity and $k$ is thermal diffusivity.

**(a)** Using standard dimensions of $T$, $L$, $\Theta$ and $E$ and this specific order of variables, show that the dimension matrix looks like

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

when $[\![t]\!] = $ T, $[\![e]\!] = $ E, $[\![r]\!] = $ L, $[\![k]\!] = L^2T^{-1}$, $[\![u]\!] = \Theta$, and $[\![c]\!] = E\Theta^{-1}L^{-3}$.

**(b)** Find the nullspace of the dimension matrix to obtain dimensionless collections of the variables $\pi_1, \pi_2, \ldots \pi_k$.

Question **How many $\pi_k$ do you expect to find and how is this related to the rank of the dimension matrix?)**

**(c)** then use dimensional analysis and the BPT (Buckingham Pi Theorem) to show that we can re-write $f(t, r, u, e, k, c) = 0$ as $F(\pi_1, \pi_2) = 0$ where

$$\pi_1 = \frac{r}{\sqrt{kt}} \text{ and } \pi_2 = \frac{uc}{e}(kt)^{3/2}$$

**(d)** Therefore

$$\frac{e}{c}(kt)^{-3/2} g\left(\frac{r}{\sqrt{kt}}\right) = u$$

is a version of a physical law relating the quantities.