Senior Colloquium: *Applied Mathematics*

Math 400 Fall 2020 `http://zoom.us/j/3232592536` T 10:15am - 11:40am

 `http://sites.oxy.edu/ron/math/400/f20/`

Class 10: Tuesday October 27

TITLE Advanced Boundary Layer Theory

CURRENT READING Logan, §3.3 (pp. 179-185); Holmes, §2.5-2.6 (pp. 69-84); Witelski, §7.1-7.3 (pp. 147-158)

SUMMARY

This week we will learn how to do determine the size of a boundary layer (dominant balancing returns!) and compute boundary layer in other locations besides the leftmost boundary.

BOUNDARY LAYER SIZE

Recall the boundary value problem from Week #9:

$$\epsilon\frac{d^2y}{dx^2} + (1+\epsilon)\frac{dy}{dx} + y = 0, \text{ where } \epsilon \ll 1 \text{ and } \quad 0 < x < 1 \text{ with } y(0) = 0, \quad y(1) = 1. \tag{1}$$

We assumed that this problem has a boundary layer because we know it is a singular perturbation problem. We assumed the boundary layer was $\mathcal{O}(\epsilon)$ and at $y = 0$. Today we shall show how to make those determinations for ourselves.

LEFT OR RIGHT?

Logan (p. 188) says that for the standard BVP

$$\epsilon y'' + p(x)y' + q(x)y = 0, 0 < x < 1, \quad y(0) = a,\ y(1) = b$$

that when $p(x) > 0$ the boundary layer is at the LEFT end, and when $p(x) < 0$ the boundary layer is at the right.

We found the outer problem for y_{outer} which is valid when ϵ is ignored, i.e. in the range where $\mathcal{O}(\epsilon) < x \leq 1$, by letting $\epsilon = 0$ in Equation (1). The outer problem becomes

$$y'_{outer} + y_{outer} = 0, \quad y_{outer}(1) = 1 \tag{2}$$

Solving the IVP in (2) gives us the solution $y_{outer}(x) = e^{1-x}$.

However, in the boundary layer (inner problem) we have to do more work. We must re-scale the independent variable x using the following:

$$\xi = \frac{x}{\delta(\epsilon)} \text{ and } Y(\xi) = y(x) = y(\xi\delta(\epsilon)) \tag{3}$$

and plug in the new variables in (3) into the original equation in (1) produces

$$\underset{①}{\frac{\epsilon}{\delta(\epsilon)^2}}\frac{d^2Y}{d\xi^2} + \left(\underset{②}{\frac{1}{\delta(\epsilon)}} + \underset{③}{\frac{\epsilon}{\delta(\epsilon)}}\right)\frac{dY}{d\xi} + \underset{④}{Y(\xi)} = 0 \tag{4}$$

There are four terms to do a dominant balancing of, $\frac{\epsilon}{\delta(\epsilon)^2}$, $\frac{1}{\delta(\epsilon)}$, $\frac{\epsilon}{\delta(\epsilon)}$ and 1.

This results in $\binom{4}{2} = 6$ possible balances.

EXAMPLE

Let's show that a consistent balancing is available if $\delta(\epsilon) = \epsilon$ is chosen.

CASE I: ① ≈ ② CASE II: ① ≈ ③ CASE III: ① ≈ ④

CASE IV: ② ≈ ③ CASE V: ② ≈ ④ CASE VI: ③ ≈ ④

Choosing the scaling $\delta(\epsilon) = \epsilon$ and plugging back into (4) leads to

$$\epsilon Y'' + Y' + \epsilon Y' + \epsilon Y = 0, \quad Y(0) = 0 \tag{5}$$

which is an ODE that can be approximated using regular perturbation, so we set $\epsilon = 0$ and solve the leading order problem $Y'' + Y' = 0$ with $Y(0) = 0$, which has the solution $y_{inner} = A(1 - e^{-x/\epsilon})$. We then using asymptotic matching to find the value of A

$$\lim_{x \to 0^+} y_{outer}(x) = \lim_{\xi \to \infty} y_{inner}(\xi) = y_{BLC} \tag{6}$$

This give us the result $y_{BLC} = e$. To summarize, we now have

$$\begin{aligned} y_{inner}(x) &= e(1 - e^{-x/\epsilon}), \text{ when } 0 \le x \le \mathcal{O}(\epsilon) \\ y_{outer}(x) &= e^{1-x}, \text{ when } \mathcal{O}(\epsilon) < x \le 1 \end{aligned}$$

Then we can write down the uniform expansion to the solution

$$y_{uniform}(x) = y_{inner}(x) + y_{outer} - y_{BLC} \tag{7}$$

So

$$y_{uniform}(x) = e^{1-x} + e(1 - e^{-x/\epsilon}) - e = e^{1-x} - e^{1-x/\epsilon}$$

BOUNDARY LAYER AT THE RIGHT END

Consider the following BVP

$$\epsilon^2 \frac{d^2y}{dx^2} - \frac{dy}{dx} + xy = 0, \text{ where } 0 < \epsilon \ll 1 \text{ and } \quad 0 < x < 1 \text{ with } y(0) = 3, \quad y(1) = -1. \quad (8)$$

Algorithm for Solving Boundary Layer Problems

STEP 0: Will Regular Perturbation Work?

STEP 1: Is there a boundary layer? What's the sign of the y' term?
(If > 0 boundary layer is at the leftmost end, if < 0 boundary layer is at rightmost end.)

STEP 2: Use dominant balancing to determine the form of $\delta(\epsilon)$ by using $\xi = \dfrac{x - x_0}{\delta}$ and $x = x_0 + \xi\delta(\epsilon)$ where x_0 is the location of the boundary layer in the interval $0 < x < 1$.

STEP 3: Is the ODE linear in y and satisfies Logan's Theorem 3.12? If yes, Use it! If not, proceed to find inner and outer problems and solve them and then use asymptotic matching to obtain a uniform expansion.

STEP 4: Check that your uniform expansion satisfies the initial conditions and ODE asymptotically (as $\epsilon \to 0$). If you know exact solution, compare it with your uniform expansion.

EXAMPLE

Let's show that $\delta(\epsilon) = \epsilon^2$ assuming the boundary layer is at the right end, i.e. $x_0 = 1$.

Exercise

Rescale the problem in Equation (8)

$$\epsilon^2 y'' - y' + xy = 0,\ y(0) = 3,\ y(1) = -1$$

using $\xi = \dfrac{x-1}{\epsilon^2}$ and show that the outer and inner problems become:

OUTER PROBLEM (at $x = 0$)

$$-y' + xy = 0,\ y(0) = 3. \tag{9}$$

INNER PROBLEM (at $x = 1$)

$$Y'' - Y' + \epsilon^2 Y + \epsilon^4 \xi Y = 0,\ Y(0) = -1. \tag{10}$$

GROUPWORK

(A) Solve the inner problem (10) and outer problem (9).

(B) Use asymptotic matching for a right-end boundary layer

$$y_{BLC} = \lim_{x \to 1^-} y_{outer} = \lim_{\xi \to -\infty} y_{inner}$$

(C) Show that the uniform expansion for the solution to Equation (8) is

$$y_{uniform} = 3 \exp\left(\frac{x^2}{2}\right) - (1 + 3\sqrt{e}) \exp\left(\frac{x-1}{\epsilon^2}\right) \tag{11}$$

and check that it satisfies boundary conditions and ODE asymptotically.

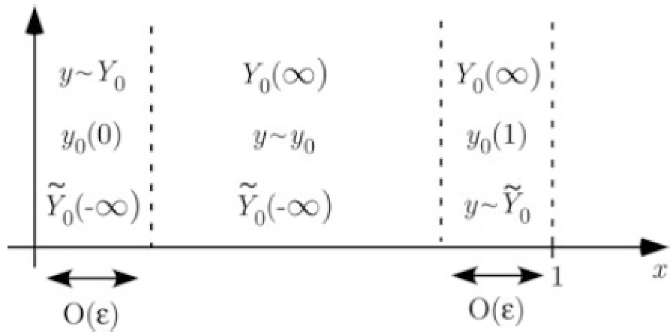
BOUNDARY LAYER AT BOTH ENDS(!)

TUTORIAL

(Adapted from Holmes, p. 81, Example 3) Consider the following BVP on $0 < x < 1$ with $0 < \epsilon \ll 1$ which has boundary layers at each end of the interval:

$$\epsilon^2 y'' + \epsilon x y' - 2y = 2 - 2x,\ y(0) = 2,\ y(1) = -1.$$

Find a composite expansion of the leading order solution. Here's a diagram of the situation from Holmes:



where y_0 is the "outer" solution and Y_0 is the expression in the left boundary layer and $\tilde{Y}_0$ is the expression in the right boundary layer.

$$y_{uniform} = y_0(x) + Y_0(\xi) + \tilde{Y}_0(\tilde{\xi}) - y_0(0) - y_0(1)$$