Senior **C**olloquium: *History of Mathematics*

Math 400 Spring 2020 | Fowler 310 T 1:30pm - 2:55pm

 | http://sites.oxy.edu/ron/math/400/20/

Worksheet 2: **Tuesday February 4**

TITLE Euler (Part 2): "Master of us all"
CURRENT READING Katz, 594-600; Boyer 405-419
NEXT READING Katz, 712, 819-822, 834-838; Boyer 464-476

Leonhard Euler (1707-1783)
Euler was born in Basel, Switzerland but spent most of his life in St. Petersburg, Russia and Berlin, Germany. He is widely regarded as the most prolific mathematician of all time, with significant contributions to various branches of mathematics. He is most well-known for his establishment of particular symbols which his use led to their universal adoption. Boyer presents a small list of these symbols we use now that are attributable to Euler:

$f(x)$	For functional notation
e	For the base of the natural logarithms
a, b, c	For the sides of the triangle ABC
s	For the semiperimeter of triangle ABC
r	For the inradius of the triangle ABC
Σ	For the summation sign
i	For the imaginary unit $\sqrt{-1}$

Calculus Results
Euler published *Institutiones calculi differentialis (Method of the Differential Calculus)* in 1755 and *Institutiones calculi integralis (Method of the Integral calculus)* in 1768.

Euler used infinite series in clever ways to prove the formula for the quotient rule and derivative of the logarithm.

First he expanded

$$\frac{1}{q+dq} = \frac{1}{q}\left(1 - \frac{dq}{q} + \frac{(dq)^2}{q^2} - \ldots\right)$$

And neglecting higher order terms leads to

$$\frac{p+dp}{q+dq} = (p+dp)\left(\frac{1}{q} - \frac{dq}{q^2}\right)$$

Which allows you to write an expression for

$$d\left(\frac{p}{q}\right) = \frac{p+dp}{q+dq} - \frac{p}{q}$$

so that it becomes

$$d\left(\frac{p}{q}\right) = \frac{qdp - pdq}{q^2}$$

Exercise

Show that when $y = \ln x$, $dy = \ln(x + dx) - \ln(x)$ so that $dy = \frac{dx}{x}$

(Recall that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$)

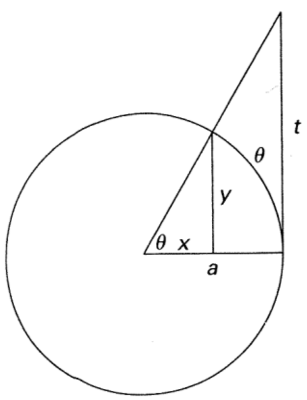
Euler Resolves Dispute Over Logarithms of Negative and Complex Numbers

D'Alembert (and other mathematicians like John Bernouilli) were convinced that log(-*x)*=log(x) because

$$(-x)^2 = x^2$$
$$\log[(-x)^2] = \log[x^2]$$
$$2\log(-x) = 2\log(x)$$
$$\log(-x) = \log(x)$$

However, Euler in a letter entitled ON THE CONTROVERSY BETWEEN Messrs. LEIBNIZ AND BERNOULLI CONCERNING LOGARITHMS OF NEGATIVE AND IMAGINARY NUMBERS to D'Alembert that was later published explained how to compute the logarithm of any negative or complex number.

Consider page 597 of Katz



Given $x=\cos\theta$ and $y=\sin\theta$ then $\theta=\arctan(y/x)=\arctan(t/a)$, so

$$\begin{aligned}\frac{a^2}{2}\theta &= \frac{a^2}{2}\arctan\frac{t}{a} \\ &= \frac{a^2}{2}\int\frac{a\,dt}{a^2+t^2} = \frac{a^2}{2}\int\left[\frac{1/2}{a+ti}+\frac{1/2}{a-ti}\right]dt \\ &= \frac{a^2}{4}\left[\frac{1}{i}\ln(a+ti)-\frac{1}{i}\ln(a-ti)\right] = \frac{a^2}{4i}\ln\left[\frac{a+ti}{a-ti}\right] \\ &= \frac{a^2}{4i}\ln\left[\frac{a+(ay/x)i}{a-(ay/x)i}\right] = \frac{a^2}{4i}\ln\left[\frac{x+yi}{x-yi}\right]\end{aligned}$$

Let $x=0$ and $\theta=\pi/2$ gives you the result that

$$\frac{\pi a^2}{4} = \frac{a^2}{4i}\ln(-1)$$

Which implies that $\ln(-1)=i\pi$. Interestingly, we can use this result to derive the "most beautiful equation in all of Mathematics": $e^{i\pi}+1=0$ (also known as **Euler's Identity**).

Exercise

Show that $e^{i\pi}+1=0$ and $\ln(-1)=i\pi$ are equivalent statements.

In *Complex Analysis* we learn that the complex logarithm function is an infinitely-valued function

$$\log z = \log(x+iy) = \ln\left(\sqrt{x^2+y^2}\right) + i(\theta+2n\pi) \qquad \text{where } \theta = \arctan(y/x) \text{ and } n \text{ is any integer}$$

The complex logarithm is often defined by its "principal branch" to make it a single-valued function and is written Log z. This involves restricting the value of θ=arctan(y/x) (called the argument function) to a range of $-\pi < \theta \leq \pi$.

$$\text{Log } z = \text{Log}(x+iy) = \ln\left(\sqrt{x^2+y^2}\right) + i\theta \text{ where } -\pi < \theta \leq \pi.$$

The Most Beautiful Equation In All Of Mathematics

Euler is very well known for the result that

$$e^{ix} = \cos(x) + i\sin(x)$$

Which of course when $x=\pi$ leads to "the most beautiful equation in all of mathematics" (also known as **Euler's Identity**)

$$e^{i\pi} + 1 = 0$$

Euler was also able to show that

$$i^i = e^{-\pi/2}$$

GROUPWORK

Try to replicate Euler's result that $i^i = e^{-\pi/2}$. Use the fact that when $z, c \in \mathbb{C}$, $\quad z^c = e^{\log(z^c)} = e^{c\log z}$.

(What happens if you don't use the principal branch of log z?)

The Many Contributions of Euler

Euler has his name on so many concepts, procedures and ideas that it has become customary to name things for the person who rediscovered them first ***after*** Euler.

Here are just some of the classic results that have survived to keep his name.

Euler's Constant

A constant that shows up in Number Theory that measures the difference between the sum of the harmonic series and the natural logarithm function as they both, diverge.

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right) = 0.57721...$$

Euler's Method

This is a method for approximating solutions to differential equations of the form

$$\frac{dy}{dt} = f(y,t), \quad y(a) = b$$

Numerically using the algorithm

$$y_{n+1} \approx y_n + f(y_n, t_n)(t_{n+1} - t_n)$$

Euler characteristic

Stemming from his solution of the famous Seven Bridges of Konigsberg problem, Euler showed that a finite, connected planar graph (with no edge intersections) has the property that there was a simple relationship between the number of **vertices V, edges E** and **faces F** such that **V-E+F=γ**. The topology of the object determine the value of the Euler characteristic **γ.** For example, for planar graphs **γ=1** but for Platonic solids **γ=2**.

Euler-Lagrange Equation

The Euler-Lagrange equation is the basic equation in the area of mathematics called the calculus of variations. Given an integral of the form $I(y) = \int_a^b F(x, y, y')dx$ the problem is to find a curve *y(x)* that minimizes or maximizes the value of *I(y).* The condition that *y(x)* must satisfy is called the Euler-Lagrange equation and looks like

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Euler Differential Equation

An Euler (ordinary differential) equation is one of the form

$$a_2 x^2 y''(x) + a_1 x y'(x) + a_0 y = f(x)$$

Where the order of the derivative and the order of the independent variable are paired. Euler showed that using the transformation $x = e^t$ one can transform this into a constant coefficient differential equation, which is then very easy to solve.

Euler-Euclid Theorem

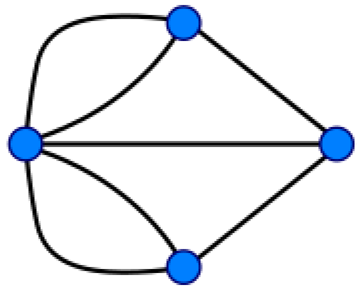
This theorem connects even perfect numbers and prime numbers. The statement of the theorem is:

Every even perfect number has the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime number.

The numbers $2^p - 1$ are known as Mersenne primes and there is a conjecture that there are an infinite number of them but only 51 have been discovered to date). A perfect number *P* is a natural number that is equal to the sum of its proper divisors (i.e. numbers that are less than *P* and divide it evenly). For example, 6 is a perfect number because its divisors are 3, 2 and 1.

GroupWork

A. Compute the number of vertices, edges and faces for the Konigsberg Bridge graph:



B. Convert the following ordinary differential equation $x^2y'' + 6xy' + 6y = 0$ into a constant coefficient ODE and find the general solution. (Check your answer!)

C. Confirm that when *p=3*, the Euler-Euclid formula produces an even perfect number and a Mersenne prime.

D. Use the method of Calculus of Variations to prove that the shortest distance between two points is a straight line. In other words, considering $F(x, y, y') = \sqrt{1 + y'^2}$ solve the Euler-Lagrange equation.