Senior Colloquium: *History of Mathematics*

Math 400 Spring 2020 | Fowler 310 T 1:30pm - 2:55pm

`http://sites.oxy.edu/ron/math/400/20/`

Homework #7

[8 points]

ASSIGNED: Tue Apr 14 2020 **DUE: Tue Apr 21 2020**

The Riemann Zeta Function and the Bernouilli Numbers

These problems will relate the Riemann Zeta Function, which is related to the Riemann Hypothesis (one of the most important unsolved problems in pure mathematics), and Bernouilli numbers, which are named after Jacob Bernouilli (1655-1705).

The Riemann Zeta function is

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

The n^{th} Bernouilli number B_n can be written in terms of ζ as

$$B_n = -n\zeta(1-n) \text{ for } n \geq 2$$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \text{ for } n = 1, 3, 5, \dots$$

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|$$

Note that all odd Bernouilli numbers after B_1 are identically zero, i.e. $B_3 = B_5 = B_7 = B_{2k+1} = 0$ for $k = 1, 2, 3, \dots$.

One can compute the Bernouilli numbers directly using the formula

$$B_n = \lim_{x \to 0} \frac{d^n}{dx^n} \left[\frac{x}{e^x - 1} \right] \tag{1}$$

Or one can obtain Bernouilli numbers by looking closely at the coefficients of Taylor expansions of certain functions. For example,

$$\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} x^{2n-1}, \qquad |x| < \frac{\pi}{2}. \tag{2}$$

1. **The famous Ramanujan sum**. *4 points.* When Ramanujan first wrote G.H. Hardy one of the results that amazed and perturbed the British mathematician was the following (nonsensical result)

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12} \tag{3}$$

   We can show where this first example of a "Ramanujan sum" comes from by using the Riemann Zeta function.

   (a) *1 point.* Show that LHS of (3) is clearly equal to $\zeta(-1)$ and the RHS is equal to $-\frac{B_2}{2}$

   (b) *2 points.* Use one of the formulas in (1) or (2) to compute B_2.

   (c) *1 point.* Discuss your interpretation of the result that you have just proved given in (3). Why (or why not) does this equation make sense?

2. **Back to Basel.** *4 points.* We previously discussed Euler's solution of the Basel problem, i.e. $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Using the Riemann Zeta function we can show how he was able to also give exact values for $\sum_{k=1}^{\infty} \frac{1}{k^4}, \sum_{k=1}^{\infty} \frac{1}{k^6}, \ldots, \sum_{k=1}^{\infty} \frac{1}{k^{2n}}$ for any value of n.

   (a) *1 point.* Use your previously computed value of B_2 to compute $\zeta(2)$ and confirm the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

   (b) *1 point.* Compute $\zeta(4)$ to find an exact value of $\sum_{k=1}^{\infty} \frac{1}{k^4}$. What Bernouilli number will you need to compute in order to obtain the answer? [HINT: Use Formula (2) to calculate this B_n.]

   (c) *2 points.* Obtain a general formula for computing the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$ like Euler did which involves B_{2n}. Use it to find the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^{18}}$. You can look up the value of the Bernouilli number you need instead of computing it this time!