Senior Colloquium: *History of Mathematics*

Math 400 Spring 2020 — Fowler 310 T 1:30pm - 2:55pm

 — `http://sites.oxy.edu/ron/math/400/20/`

Homework #5

[5 points]

ASSIGNED: Tue Feb 25 2020 **DUE: Tue Mar 3 2020**

Cauchy's Residue Calculus

We will look at two applications of Cauchy's Residue Calculus that assist us in computing the value of two real integrals. Recall the definition of a residue of a pole of order m of a function $f(z)$ at z_0 is given by

$$\textbf{Res}(f; z_0) = \lim_{z \to z_0} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right\}$$

1. **Trigonometric Integrals** We want to show that $\mathcal{I} = \displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}$.

   (a) *1 point.* Show that if $z = e^{i\theta} = \cos\theta + i\sin\theta$ and $\sin\theta = \dfrac{z - 1/z}{2i}$, $|z| = 1$ and $dz = iz\, d\theta$ then $\mathcal{I} = \displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$ can be re-written as $\mathcal{I} = \displaystyle\oint_{|z|=1} \frac{2dz}{4z^2 + 10iz - 4}$.

   (b) *2 points.* Use Cauchy's Residue Theorem after finding the *relevant* poles and residues of $f(z) = \dfrac{2}{4z^2 + 10iz - 4}$ to show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}$. Explain your answer and show all your work.

2. **Improper Integrals** We want to show that $\mathcal{J} = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2} = \frac{\pi}{2}$.

   (a) *1 point.* Show that the expression $f(z) = \dfrac{1}{(1 + z^2)^2}$ has poles of order 2 at $z = \pm i$ with residue equal to $\mp \dfrac{i}{4}$.

   (b) *1 point.* Use Cauchy's Residue Theorem to show that $\mathcal{J} = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2} = \frac{\pi}{2}$. Explain your answer and show all your work.