# HW #5

(1pt) 1 (a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_{|z|=1} \frac{dz}{iz}\, \frac{1}{\left[5+4\left(z-\frac{1}{z}\right)\frac{1}{2i}\right]} = I$

$z = e^{i\theta}$ $\qquad \sin\theta = \left(z-\frac{1}{z}\right)\frac{1}{2i}$

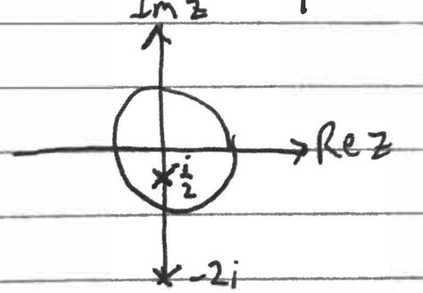
$dz = ie^{i\theta}d\theta = iz\,d\theta$

$$I = \int_{|z|=1} \frac{1}{iz}\frac{dz}{\left[5+\left(\frac{z^2-1}{z}\right)(-2i)\right]} = \int_{|z|=1} \frac{dz}{5z-2iz^2+2i}\,\frac{1}{i}$$

$$= \int_{|z|=1} \frac{dz}{2z^2+5iz-2} = \int_{|z|=1} \frac{2\,dz}{4z^2+10iz-4}$$

(2pt) (b) $2z^2+5iz-2=0 \Rightarrow z = \dfrac{-5i \pm \sqrt{(5i)^2-4\cdot 2\cdot(-2)}}{2(2)}$

$$z = \frac{-5i \pm \sqrt{-25+16}}{4} = \frac{-5i \pm \sqrt{-9}}{4} = \frac{-5i \pm 3i}{4} = -2i \text{ or } -\frac{i}{2}$$

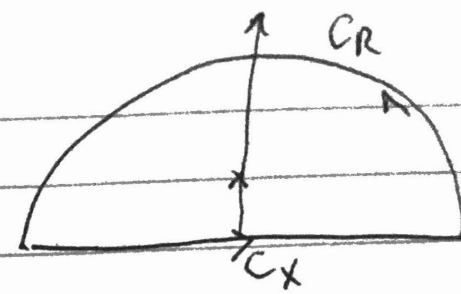



$$\mathrm{Res}\left(-\frac{i}{2}, \frac{1}{2z^2+5iz-2}\right) = \lim_{z\to -\frac{i}{2}} \frac{(z - -i/2)}{2z^2+5iz-2}$$

$$= \lim_{z\to -i/2} \frac{1}{4z+5i} = \frac{1}{4\left(-\frac{i}{2}\right)+5i} = \frac{1}{3i}$$

$$I = 2\pi i\, \mathrm{Res}\left(-\frac{i}{2}, \frac{1}{2z^2+5iz-2}\right) = 2\pi i \cdot \frac{1}{3i} = \frac{2\pi}{3}$$

(1pt) 2(a) $J = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$




$$\int_{C_R} \frac{dz}{(1+z^2)^2} = 2\pi i \, \mathrm{Res}\left(i, \frac{1}{(1+z^2)^2}, 2\right)$$

$$(z^2+1)^2 = 0 \Rightarrow z = \pm i$$

$$\lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{1}{(z-i)^2(z+i)^2}\right] = \lim_{z\to i} \frac{d}{dz} \frac{1}{(z+i)^2} = \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{-2}{(2$$

$$= \frac{-2}{-8i} = \frac{-i}{4}$$

$$\lim_{z\to -i} \frac{d}{dz}\left[(z+i)^2 \frac{1}{(z-i)^2(z+i)^2}\right] = \lim_{z\to -i} \frac{d}{dz} \frac{1}{(z-i)^2} = \lim_{z\to -i} \frac{-2}{(z-i)^3} = \frac{-2}{(-2i}$$

$$= \frac{-2}{8i} = \frac{i}{4}$$

(1pt) 2b $J = 2\pi i\left(\frac{-i}{4}\right) = \frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \int_{C_x} \frac{dz}{(1+z^2)^2} + \int_{C_R} \frac{dz}{(1+z^2)}$

$$= 2\pi i \, \mathrm{Res}\left(i, \frac{1}{(1+z^2)^2}\right)$$

$$\int_{C_R} \frac{dz}{(1+z^2)^2} \to 0 \text{ as } R \to \infty$$