# HW #4

+1 (a) $\displaystyle\int_1^3 1 - \frac{1}{x^2}\,dx = \left(x + \frac{1}{x}\right)\Big|_1^3 = \left(3 + \frac{1}{3}\right) - \left(1 + \frac{1}{1}\right) = \frac{10}{3} - 2 = \frac{4}{3}$

+1 (b) An affine transformation from $1 \le x \le 3$ to $-1 \le u \le 1$

$u = x - 2$

In general a transformation from $[a, b]$ to $[-1, 1]$ is

$x = a,\ u = -1$ $\quad \cancel{u = c_1 x + c_2 \Rightarrow a = c_1(-1) + c_2 \Rightarrow c_2 = (a+b)/2}$

$x = b,\ u = 1$ $\quad u = \left(\frac{b-a}{2}\right)x + \left(\frac{b+a}{2}\right)$ $\cancel{b = c_1(1) + c_2 \quad c_1 = (b-a)/2}$

$\cancel{u - \left(\frac{b+a}{2}\right) = \left(\frac{b-a}{2}\right)x \Rightarrow x = u\left(\frac{a}{b-a}\right) - \frac{b+a}{b-a}}$

$u = c_1 x + c_2$ $\qquad -1 = \frac{2}{b-a}a + c_2 \Rightarrow c_2 = -1 - \frac{2a}{b-a}$

$-1 = c_1 a + c_2$

$1 = c_1 b + c_2$ $\qquad c_2 = \frac{-b-a}{b-a} \Leftarrow c_2 = \frac{-b+a-2a}{b-a}$

$-2 = c_1(a-b) \Rightarrow c_1 = \frac{2}{b-a}$ $\qquad c_2 = \frac{b+a}{a-b}$

$$\boxed{u = \frac{2}{b-a}x + \frac{a+b}{a-b}}$$

$u = x - 2$

$du = dx$

+1 (c) $\displaystyle I = \int_1^3 1 - \frac{1}{x^2}\,dx = \int_{-1}^{1} 1 - \frac{1}{(u+2)^2}\,du = \int_{-1}^{1} f(u)\,du$

$$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 1 - \frac{1}{\left(-\frac{1}{\sqrt{3}} + 2\right)^2} + 1 - \frac{1}{\left(\frac{1}{\sqrt{3}} + 2\right)^2}$$

$$= 1 - \frac{3}{(2\sqrt{3} - 1)^2} + 1 - \frac{3}{(2\sqrt{3}+1)^2}$$

$$= 2 - 3\left[\frac{1}{(2\sqrt{3}-1)^2} + \frac{1}{(2\sqrt{3}+1)^2}\right] = 2 - 3\left[\frac{(2\sqrt{3}+1)^2 + (2\sqrt{3}-1)^2}{(2\sqrt{3}-1)^2(2\sqrt{3}+1)^2}\right]$$

# HW4

$$I = 2 - 3\left[\frac{(2\sqrt{3})^2 + 1^2 + 2\sqrt{3}\cdot 2 + (2\sqrt{3})^2 + 1^2 - 2\sqrt{3}\cdot 2}{[(2\sqrt{3})^2 - 1^2][(2\sqrt{3})^2 - 1^2]}\right]$$

$$= 2 - 3\left[\frac{12+1+12+1}{(12-1)(12-1)}\right] = 2 - 3\left(\frac{26}{121}\right) = \frac{242-78}{121}$$

$$I = \frac{164}{121} \approx 1.3553719$$

We would not expect the answer to be exact because $f(x)$ being integrated is NOT a polynomial of degree 3 or less.

+2 (d) $f(u) = 1 - \frac{1}{(2+u)^2}$ on $-1 \le u \le 1$

$N = 2$

$f' = \frac{2}{(2+u)^3}$ $f'' = -\frac{6}{(2+u)^4}$ $f''' = \frac{24}{(2+u)^5}$ $f^{iv} = \frac{-120}{(2+u)^6}$

$$\text{Error} \approx \frac{2^{2\cdot(2)+1}(2!)^4}{[2(2)+1][2\cdot(2)!]^3} f^{(4)}(\xi) \approx \frac{2^5\, 2^4}{5\cdot(4!)^3} f^{(4)}(\xi)$$

$$\approx \frac{16\cdot 32}{5\cdot(24)^3} f^{(4)}(\xi)$$

$$\approx \frac{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot 2}{5\cdot\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot 6\cdot 6\cdot 6} f^{4}(\xi)$$

$$\approx \frac{1}{5\cdot 3\cdot 7\cdot 3} f^{(4)}(\xi) \approx \frac{1}{105} f^{(4)}(\xi)$$

$$\approx \frac{120}{105} \approx \frac{8}{9}$$ upper bound

Because error term has $f^{(2N)}(\xi)$ term when a polynomial of degree $2N-1$ is approximated, Gaussian Quadrature is EXACT!

Actual error was $\frac{4}{3} - \frac{164}{121} \approx \frac{484-492}{363} \approx \frac{-8}{363}$