Senior **C**olloquium: *History of Mathematics*

Math 400 Spring 2020 Fowler 310 T 1:30pm - 2:55pm

http://sites.oxy.edu/ron/math/400/20/

Homework #3

[8 points]

1. Let's consider another set of orthogonal polynomials, the Chebyshev Polynomials of the First Kind $T_n(x)$. Note that the first two polynomials $T_0(x)$ and $T_1(x)$ are identical to the first two Legendre polynomials but then the polynomials differ...

$$T_0(x) = 1 \quad (1)$$
$$T_1(x) = x \quad (2)$$
$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad (3)$$

   (a) *1 point.* Find $T_2(x)$ and $T_3(x)$.

   (b) *2 points.* The Chebyshev differential equation for $T_n(x)$ is

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0.$$

   Show that T_0, T_1, T_2 and T_3 satisfy this ODE.

   (c) *2 points.* The Chebyshev polynomials of the first kind satisfy the property that $T_n(\cos x) = \cos(nx)$. Use this definition (and trigonometric identities) to verify the recursion formula $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

   (d) *3 points.* Like the Legendre polynomials, the Chebyshev polynomials of the first kind are also orthogonal on L^2 over [-1,1] but this time the weight function is $(1-x^2)^{-1/2}$. Show that

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n \neq 0 \\ \pi, & \text{if } m = n = 0 \end{cases}$$

   (HINT: use a trigonometric substitution to simplify the integral and $T_k(x)$....)