HW#3
[8 pts]

1. (a) $T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2xT_1 - T_0 = 2x(x) - 1 = 2x^2 - 1$

$T_3(x) = 2xT_2 - T_1 = 2x(2x^2-1) - x = 4x^3 - 2x - x$

$= 4x^3 - 3x$

(b) $(1-x^2)y'' - xy' + n^2y = 0$

$n=0,\ T_0 = 1,\ T_0' = 0,\ T_0'' = 0$

$(1-x^2)\cdot 0 - x\cdot 0 + 0^2\cdot 1 = LHS = 0 = RHS$

$n=1,\ T_1 = x,\ T_1' = 1,\ T_1'' = 0$

$(1-x^2)\cdot 0 - x\cdot 1 + 1^2\cdot x = LHS = 0$

$-x+x = 0 = RHS$

$n=2,\ T_2 = 2x^2-1,\ T_2' = 4x,\ T_2'' = 4$

$(1-x^2)\cdot 4 - x\cdot 4x + 2^2(2x^2-1) = 4 - 4x^2 - 4x^2 + 8x^2 - 4$

$= 0 = LHS = RHS$

$n=3,\ T_3 = 4x^3-3x,\ T_3' = 12x^2-3,\ T_3'' = 24x$

$(1-x^2)24x - x\cdot(12x^2-3) + 3^2(4x^3-3x) = LHS$

$24x - 24x^3 - 12x^3 + 3x + 36x^3 - 27x =$

$0 = LHS = RHS$

(c) $T_n(\cos x) = \cos(nx)$

$T_{n+1}(\cos x) = 2(\cos x)T_n(\cos x) - T_{n-1}(\cos x)$

$= 2\cos x\cos(nx) - \cos(n-1)x$

$\cos(n+1)x = 2\cos x\cos(nx) - \cos(n-1)x$

$\cos(n+1)x + \cos(n-1)x = 2\cos x\cos(nx)$

Recall $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$LHS = \cos(n+1)x + \cos(n-1)x = 2\cos(nx)\cos(x) = RHS$ ✓

HW #3

$$I = \int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx = \int_{\pi}^{0} \cos(n\theta) \frac{\cos(m\theta)}{\sin\theta} (-\sin\theta) d\theta$$

Let $x = \cos\theta$ $\quad dx = -\sin\theta \, d\theta$

$x = -1, \; \theta = \pi$ $\quad T_n(\cos\theta) = \cos n\theta$

$x = 1, \; \theta = 0$ $\quad T_m(\cos\theta) = \cos m\theta$

$$I = \int_0^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta$$

If $n = m = 0$, $I(0,0) = \int_0^{\pi} 1 \, d\theta = \pi$

If $m = n \neq 0$ $\quad I(m,m) = \int_0^{\pi} \cos^2(m\theta) d\theta = \int_0^{\pi} \frac{\cos(2m\theta)+1}{2} d\theta$

Recall $\cos(2m\theta) = 2\cos^2(m\theta) - 1 \iff \cos^2(m\theta) = \frac{\cos(2m\theta)+1}{2}$

$$I(m,m) = \frac{1}{2}\left(\frac{\sin 2m\theta}{2m} + \theta\right)\Big|_0^{\pi} = \frac{\pi}{2} \checkmark$$

If $m \neq n$

$$I(m,n) = \frac{1}{2}\int_0^{\pi} \cos(m+n)\theta + \cos(m-n)\theta \, d\theta = \frac{1}{2}\frac{\sin(m+n)\theta}{m+n} + \frac{1}{2}\frac{\sin(m-n)\theta}{m-n}\Big|_0^{\pi}$$

$$= \frac{1}{2(m+n)}\sin(m+n)\pi + \frac{1}{2(m-n)}\sin(m-n)\pi - 0$$

$$= 0$$

$$I(m,n) = \begin{cases} 0, & m \neq n \\ \pi, & m = n = 0 \\ \frac{\pi}{2}, & m = n \neq 0 \end{cases}$$