PSP #1: Basel Problem

TASK #1 $a_{k+1} = r a_k$

$a_1 = r a_0$

$a_2 = r a_1 = r^2 a_0$

Geometric Series if the proportion is constant r

TASK #2 (a)

$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$ because $3 < 4$ so $\frac{1}{3} > \frac{1}{4}$

So $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$

2(b) $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$ since $5<8, 6<8, 7<8$

so $\frac{1}{5} > \frac{1}{8}, \frac{1}{6} > \frac{1}{8}, \frac{1}{7} > \frac{1}{8}$

2(c) $\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$

Since 9, 10, 11, 12, 13, 14, 15 are all less than 16, reciprocals are $> \frac{1}{16}$

2(d) $\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} > \frac{1}{2}$ Between 2^n+1 and $2^{n+1} = 2^n + 2^n = 2 \cdot 2^n$ are $2^n - 1$ terms

$2^n + 1 \le k \le 2^{n+1}$ so $\frac{1}{k} > \frac{1}{2^{n+1}}$ when $k < 2^{n+1}$

$$\sum_{k=2^n+1}^{2^n+2^n} \frac{1}{k} > \sum_{k=2^n+1}^{2^n+2^n} \frac{1}{2^{n+1}} = 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}$$

2(e) Want to show that $\sum_{k=1}^{2^n} \frac{1}{k} > \frac{n}{2}$

Use induction. When $n=1$ $\sum_{k=1}^{2} \frac{1}{k} = 1 + \frac{1}{2} = \frac{3}{2} > \frac{2}{2}$ ✓ BASE STEP

PSP #1

INDUCTIVE STEP: Assume $\sum_{k=1}^{2^n} \frac{1}{k} > \frac{n}{2}$ and show $\sum_{k=1}^{2^{n+1}} \frac{1}{k} > \frac{n+1}{2}$

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} = \underbrace{\sum_{k=1}^{2^n} \frac{1}{k}}_{\text{from } n^{th} \text{ case}} + \underbrace{\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}}_{\text{from (d)}} > \frac{n}{2} + \frac{1}{2} = \frac{(n+1)}{2}$$

STEP 2f Basically part (e) is the modern representation of Oresme's statement. If "each of which is longer than half a foot" Is the equivalent of the result from (d) and then when he says "there exist infinitely many parts.... the total will be infinite" IS equivalent to

$$\lim_{n\to\infty} A_{2^n} = \lim_{n\to\infty} \sum_{k=1}^{2^n} \frac{1}{k} \geq \lim_{n\to\infty} \frac{n}{2} = \infty$$

STEP 2g Integral Test has the same result (Harmonic Series diverges) $\int_1^\infty \frac{1}{k}\,dk = \ln(k)\Big|_1^\infty = \lim_{b\to\infty} \ln b = \infty$

Integral Div $\Longleftrightarrow$ Series diverges

TASK 3 $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{k=1}^{\infty} \frac{1}{k^2}$

Integral Test

$$\int_1^\infty \frac{1}{k^2}\,dk = \lim_{b\to\infty} \int_1^b \frac{1}{k^2}\,dk = \lim_{b\to\infty} \frac{-1}{k}\Big|_1^b = \lim_{b\to\infty} 1 - \frac{1}{b} = 1$$

Integral converges $\Longleftrightarrow$ Series converges

PSP #1

$\frac{3}{4}$

TASK #4 The famous constant is π which is involved in the quadrature of the cirle as $\pi \cdot (\text{radius})^2$.

TASK #5 5(a) $f(s) = \frac{\sin s}{s}$. The domain D of f is $D = \{s \in \mathbb{R} \setminus \{0\}\}$

5(b) f is discontinuos at $s=0$ since $f(0)$ is undefined
There are horizontal asymptotes as $s \to \pm\infty$, $f \to 0$. No vertical.

5(c) $\sin s = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \frac{s^9}{9!} - \ldots$

$$\frac{\sin s}{s} = 1 - \frac{s^2}{3!} + \frac{s^4}{5!} - \frac{s^6}{7!} + \frac{s^8}{9!} - \ldots$$

$$\lim_{s\to 0} \frac{\sin s}{s} = \lim_{s \to 0}\left(1 - \frac{s^2}{3!} + \frac{s^4}{5!} - \ldots\right) = 1$$

TASK #6 6(a) Clearly Euler was talking about $\pi = p$.
(b) Perimeter of a circle of diameter $d=1$ is $\pi d = \pi$
(c) Zeroes of this function equal the zeroes of $\frac{\sin s}{s}$ which are $\pm\pi, \pm 2\pi, \pm 3\pi$, etc
(d) 0 is not in the domain of $\sin(s)/s$.

TASK #7 (a) $x^2 - x - 6 = (x-3)(x+2) = (x-r_1)(x-r_2)$ so $r_1 = 3, r_2 = -2$
$1 + \frac{1}{6}x - \frac{1}{6}x^2 = \left(1-\frac{x}{3}\right)\left(1+\frac{x}{2}\right) = 0$ has roots $x=3$ and $x=-2$

$$\left(1-\frac{x}{r_1}\right)\left(1-\frac{x}{r_2}\right) = 1 - \frac{x}{r_1} - \frac{x}{r_2} + \frac{x^2}{r_1 r_2} = \left(\frac{r_1 r_2}{r_1 r_2} - \frac{(r_1+r_2)x}{r_1 r_2} + \frac{x^2}{r_1 r_2}\right) =$$

$$\left(1-\frac{x}{3}\right)\left(1+\frac{x}{2}\right) = 1 - \frac{x}{3} + \frac{x}{2} - \frac{x^2}{6} = 1 + \frac{1}{6}x - \frac{x^2}{6} \checkmark$$

$$= \frac{1}{r_1 r_2}\left(x^2 - (r_1+r_2)x + r_1 r_2\right)$$
$$= \frac{1}{r_1 r_2}(x-r_1)(x-r_2)$$

TASK #8 (a) Let's use the Funky Factorization Theorem since function starts with constant 1

$$1-\frac{s^2}{1.2.3}+\frac{s^4}{1.2.3.4.5}-\frac{s^6}{1.2.3.4.5.6.7}+\ldots$$

(b)

$$=\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\ldots$$

TASK #9 9(a) Use the fact $(1-a)(1+a)=(1-a^2)$ to write 8(b) as

$$\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{(2\pi)^2}\right)\left(1-\frac{x^2}{(3\pi)^2}\right)\ldots$$

9(b) Clearly the coefficient of x^2 will have the form

$$-x^2\left(\frac{1}{\pi^2}+\frac{1}{(2\pi)^2}+\frac{1}{(3\pi)^2}+\ldots\right)$$

TASK #10 (a) $1-\frac{s^2}{3!}+\frac{s^4}{5!}-\frac{s^6}{7!}+\ldots=\left(1-\frac{s^2}{\pi^2}\right)\left(1-\frac{s^2}{4\pi^2}\right)\left(1-\frac{s^2}{9\pi^2}\right)\ldots$

There are no "s" terms (linear in s) on either side

In (s^2)

(b) $LHS=-\frac{1}{3!}$ $RHS=-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\ldots\right)$

(c) $$\frac{1}{3!}=\sum_{K=1}^{\infty}\frac{1}{K^2}\cdot\frac{1}{\pi^2}$$

$$\frac{\pi^2}{3!}=\sum_{K=1}^{\infty}\frac{1}{K^2}$$

TASK #11 Euler says it is doubtful that we can discover anything new about infinite series of reciprocal powers of integers but we don't even know whether $\sum_{K=1}^{\infty}\frac{1}{K^3}$ is transcedental!