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Worksheet 13

TITLE The Calculus of Newton and Leibniz
CURRENT READING: Katz §16 (pp. 543-575) Boyer & Merzbach (pp 358-372, 382-389)

SUMMARY
We will consider the life and work of the co-inventors of the Calculus: Isaac Newton and Gottfried Leibniz.

NEXT: Who Invented Calculus? Newton v. Leibniz: Math's Most Famous *Prioritätsstreit*

Isaac Newton (1642-1727) was born on Christmas Day, 1642 some 100 miles north of London in the town of Woolsthorpe. In 1661 he started to attend Trinity College, Cambridge and assisted Isaac Barrow in the editing of the latter's *Geometrical Lectures*.

For parts of 1665 and 1666, the college was closed due to The Plague and Newton went home and studied by himself, leading to what some observers have called "the most productive period of mathematical discover ever reported" (Boyer, *A History of Mathematics*, 430).

During this period Newton made four monumental discoveries
1. the binomial theorem
2. the calculus
3. the law of gravitation
4. the nature of colors

Newton is widely regarded as the most influential scientist of all-time even ahead of Albert Einstein, who demonstrated the flaws in Newtonian mechanics two centuries later. Newton's epic work *Philosophiæ Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy)* more often known as the *Principia* was first published in 1687 and revolutionalized science itself.

Infinite Series
One of the key tools Newton used for many of his mathematical discoveries were power series.
He was able to show that he could expand the expression $(a+bx)^n$ for any n

$$(a+bx)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}bx + \binom{n}{2}a^{n-2}b^2x^2 + \binom{n}{3}a^{n-3}b^3x^3 + \ldots + \binom{n}{n}b^nx^n$$

He was able to check his intuition that this result must be true by showing that the power series for $(1+x)^{-1}$ could be used to compute a power series for log(1+x) which he then used to calculate the logarithms of several numbers close to 1 to over 50 decimal paces.

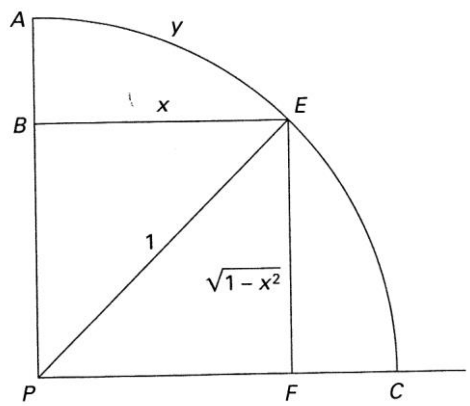
Newton's *De analysi per aequationes numero terminorum infinitas (On Analysis by Equations with Infinitely Many Terms)* was written in 1669 but was not published, although it was circulated to several of his friends and collaborators. In it, he showed that

$$y = \arcsin x = 2\int_0^x \sqrt{1-x^2}\,dx - x\sqrt{1-x^2} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$$

Since

$$\sqrt{1-x^2} = (1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \cdots$$

As demonstrated by the figure below



Newton used these results to obtain the power series expansion for the sine function, and the cosine function, writing

$$\sin y = y - \frac{1}{6}y^3 + \frac{1}{120}y^5 - \frac{1}{5040}y^7 + \ldots$$

And using $\cos y = \sqrt{1-\sin^2 y}$ to obtain an equivalent expression for cos y.

Exercise

Obtain the series of cos *y* from the power series for sine from the series for sin *y* the way Newton did it.

Newton's Calculus: Fluxions and Fluents

Newton defined a **fluxion** $\dot{x}$ as the speed of a quantity x (which he called the **fluent**) that depended on time. Newton basically thought of all derivatives as time derivatives.

He defined the **moment** of a fluent to be the amount it increases in an "infinitely small" amount of time o. Thus the moment of the fluent x is $x+\dot{x}o$.

In*Tractatus de methodis serierum et fluxionum (A Treatise on the Method of Series and Fluxions)* of 1671 Newton says:

> "[A]n equation which expresses a relationship of fluent quantities without variance at all times will express that relationship equally between $x+\dot{x}o$ and $y+\dot{y}o$ as between x and y; so $x+\dot{x}o$ and $y+\dot{y}o$ may be substituted in place of the later quantities, x and y, in the said equation."

The example Newton used was the equation $x^3 - ax^2 + axy - y^3 = 0$ which he was able to get the expression $3x^2\dot{x} - 2ax\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0$

Exercise

Considering $f(x,y)=0$ where x and y are both functions of t, obtain an expression for $\frac{df}{dt}$ in general (using the Chain Rule) and then consider $f(x,y) = x^3 - ax^2 + axy - y^3 = 0$ to replicate Newton's result.

Example

Let's use Newton's fluxion and moments to obtain the derivative of $f(x,y) = x^3 - ax^2 + axy - y^3 = 0$ with respect to time.

Finding Fluents from Fluxions

Given $\dot{y}^2 = \dot{x}\dot{y} + x^2\dot{x}^2$ Newton would re-write the expression as $\frac{\dot{y}^2}{\dot{x}^2} = x^2 + \frac{\dot{y}}{\dot{x}}$ which is a quadratic expression in $\frac{\dot{y}}{\dot{x}}$ and can be solved using the quadratic formula, obtaining

$$\frac{\dot{y}}{\dot{x}} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + x^2}$$

Which he could then use the binomial theorem on to obtain

$$y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 + \dots \text{ and } y = -\frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{2}{7}x^7 + \dots$$

GroupWork

Let's try and replicate Newton's result by using the binomial theorem. Using modern methods, what kind of equation and solution techniques would typically be used instead?

Newton's Table of Integrals

Of course, using this method to obtain fluents from fluxions would be time consuming so Newton generated a table of expressions for fluents (on the right) given the fluxion (on the left). He knew that the expression on the right could be used to obtain the area under the function on the left.

$$y = \frac{ax^{n-1}}{(b+cx^n)^2} \qquad z = \frac{(a/nb)x^n}{b+cx^n}$$

$$y = ax^{n-1}\sqrt{b+cx^n} \qquad z = \frac{2a}{3nc}(b+cx^n)^{3/2}$$

$$y = ax^{2n-1}\sqrt{b+cx^n} \qquad z = \frac{2a}{nc}\left(-\frac{2}{15}\frac{b}{c} + \frac{1}{5}x^n\right)(b+cx^n)^{3/2}$$

$$y = \frac{ax^{2n-1}}{\sqrt{b+cx^n}} \qquad z = \frac{2a}{nc}\left(-\frac{2}{3}\frac{b}{c} + \frac{1}{3}x^n\right)\sqrt{b+cx^n}$$

Gottfried Wilhelm Leibniz (1646-1716) is often called a polymath (i.e. universal genius) of the first order for his intellectual contributions to the fields of mathematics, philosophy, physics, theology, ethics and logic. He is widely regarded as a second, independent developer of differential and integral calculus as well as the primary inventor of the modern binary number system.

Leibniz was very conscious of the significance of notation and naming. It is his symbols that have mostly persisted in modern representations of Calculus, such as *dx* and ∫. Leibniz also coined the terms differential calculus and integral calculus.

The Harmonic Triangle

Leibniz was mainly self-taught in mathematics and thus often rediscovered on his own previously known results. He began his foray into discovering Calculus by noticing patterns in the sums of differences, particularly in Pascal's Triangle and in a new structure he called "the harmonic triangle."

Arithmetic triangle

1	1	1	1	1	1	1 ⋯
1	2	3	4	5	6 ⋯	
1	3	6	10	15 ⋯		
1	4	10	20 ⋯			
1	5	15 ⋯				
1	6 ⋯					
1 ⋯						

Harmonic triangle

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}\cdots$
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{30}\cdots$	
$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{30}$	$\frac{1}{60}\cdots$		
$\frac{1}{4}$	$\frac{1}{20}$	$\frac{1}{60}\cdots$			
$\frac{1}{5}$	$\frac{1}{30}\cdots$				
$\frac{1}{6}\cdots$					

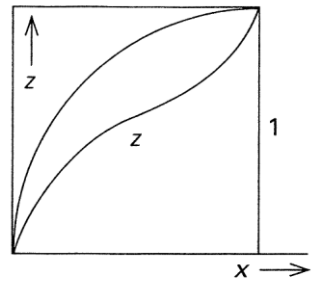
In the arithmetic triangle each element (which is not in the first column) is the difference of the two terms directly below it and to the left; in the harmonic triangle each term (which is not in the first row) is the difference of the two terms directly above it and to the right. Moreover, in the arithmetic triangle each element (not in the first row or column) is the sum of all of the terms in the line above it and to the left, whereas in the harmonic triangle each element is the sum of all of the terms in the line below it and to the right.

Leibniz was able to realize this discrete situation could be replicated in the continuous context of an infinite number of ordinates representing points on a curve.

$$\sum_{i=1}^{n} \delta y_i = y_n - y_0 \qquad \int dy = y$$

$$\delta \sum_{i=1}^{n} y_i = y_n - y_0 \qquad d\int y = y$$

Leibniz' Quadrature of the Circle



The area of a quarter of a unit circle can be represented by the following integral

$$\int y\ dx = 1 - \int \frac{z^2}{1+z^2} dz$$

But Leibniz (similar to Newton) was able to use infinite series as tools and thus approximated $(1+z^2)^{-1}$ to obtain his first famous result

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Derivative Rules

In *"A New Method for Maxima and Minima as well as Tangents, which is neither impeded by fractional or irrational Quantities and a Remarkable type of Calculus for them"* of 1684 showed how to work with differentials, which produces rules that look very much like our modern Calculus.

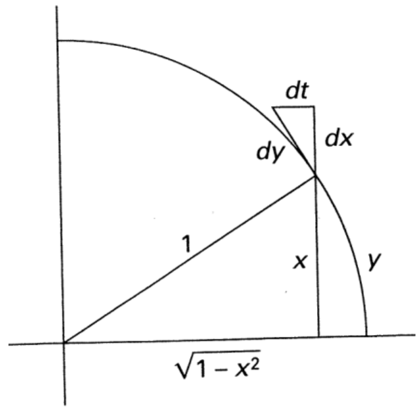
EXAMPLE

Let's show the Product Rule *d(xy)=y dx + x dy* and Quotient Rule *d(x/y)=* $\frac{ydx - xdy}{y^2}$

using differentials.

Leibniz' Approximation of the Sine Function

Consider the following diagram



It turns out that since the two right-angle triangles are similar to each other

$dt = \frac{xdx}{\sqrt{1-x^2}}$ and $\frac{dy}{1} = \frac{dt}{x}$

And by Pythagoras' Theorem $(dy)^2 = (dx)^2 + (dt)^2$

Together we can combine these results to show that $(dy)^2 = (dx)^2 + x^2(dy)^2$

Treating *dy* as a constant and applying the differential operator *d* to both sides produces

$$0 = d[(dx)^2 + x^2(dy)^2]$$
$$0 = 2d(dx)dx \ + 2xdx(dy)^2$$
$$0 = d(dx) + x(dy)^2$$
$$-x = \frac{d(dx)}{(dy)^2}$$

Leibniz solves this differential equation by assuming the solution is a power series of the form $x = a + by + cy^3 + dy^5 + ey^7 + \ldots$ where $x(0)=0$, plugging this form into both sides of the equations and equating like terms produces the equations

$$\begin{aligned} 0 &= -a \\ 2\cdot 3c &= -b \\ 4\cdot 5d &= -c \\ 6\cdot 7e &= -d \\ 8\cdot 9f &= -e \end{aligned}$$

Which when solved leads to expressions for all the terms in the series in terms of an unknown constant *b* (which one can factor out of the entire expression) or determine the unknown value of *b* by using the condition that $x'(0)=1$.

$$x = \sin(y) = y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + ...$$

GroupWork

Use Leibniz' series technique to confirm the solution *x(y)* is an approximation of the sine function given above using the modern notation for the initial value problem:

$$\frac{d^2x}{dy^2} = -x, \quad x(0) = 0, \quad x'(0) = 1$$