Worksheet 9

TITLE Mathematics of the Middle Ages and early Renaissance period
CURRENT READING: Katz §10 (pp 324-363); Boyer & Merzbach, §12-13 (pp 223-281)

SUMMARY
We will look at the medieval period of the Middles Ages (sometimes called "the Dark Ages") between the fall of Rome (476 CE) to the Renaissance (the middle of the15th century), roughly 1000 years.

NEXT: Non-Western Perspectives, Part 1 (Mayan, Indian and Islamic Mathematics)
NEXT READING: Katz §8-9 (pp 230-321) and Boyer & Merzbach, §10-11 (pp 186-222)

The *quadrivium* and the seven liberal arts
One of the only concepts that survived the Dark Ages was the idea that to be a civilized, educated man one needed to learn the quadrivium (arithmetic, geometry, music and astronomy). Boethius (480-524) added to this a trivium (grammar, rhetoric and logic) which together are known as the liberal arts.

Translations
There was almost no mathematical texts available in Europe during this time, but eventually the classic works of the Euclid's *Elements*, Ptolemy's *Almagest*, Archimedes' *Measurement of the Circle* and al-Khwarizmi's *Algebra* and *Arithmetic* were translated into Latin (see Katz, p. 327).

Leonardo of Pisa also known as **Fibonacci (**c. 1170-1240**)**
Liber abbaci (Book of Calculation) was first published in 1202. In it, Leonardo demonstrates the algorithms for how to use the Hindu-Arabic numerals and strongly advocates that fellow merchants like him adopt their use and abandon Roman numerals. *Liber abbaci* discusses "the nine Indian figures" and the sign 0, "which is called *zephirium* in Arabic." It's from this word that we get the word zero.

Note that the title of Leonardo's book had nothing to do with the calculating device known as an abacus. However, later in the 14th century people who did calculations were called *maestri d'abbaco* or abacists.

Fibonacci sequence
The most famous problem in *Liber abbaci* is often called "the rabbit problem":

> How many pairs of rabbits can be bred in one year from one pair? A certain person places one pair of rabbits in a certain place surrounded on all sides by a wall. We want to know how many pairs can be bred from that pair in one year, assuming it is their nature that each month they give birth to another pair, and in the second month after birth, each new pair can also breed.

Leonardo gave the sequence **1, 2, 3, 5, 8, 13, 21 34, 55, 89, 144, 233, 377**

Let's try to solve some of the other classic problems of Leonardo

EXAMPLE

"Suppose a lion can eat a sheep in 4 hours, a leopard can eat a sheep in 5 hours, and a bear can eat a sheep in 6 hours. How long will it take the three animals to eat one sheep together?"

Answer: $\mathbf{1\frac{23}{37}}$ **hours.**

Exercise

"Two men have some money. The first says to the second: 'If you give me one *denarius,* we will each have the same amount.' The second says to the first, 'If you give me one *denarius,* I will have ten times as much as you.' How much does each have?"

Answer: $\mathbf{1\frac{4}{9}}$ **denarius and** $\mathbf{3\frac{4}{9}}$ **denarius.**

GroupWork

"Suppose there are four men such that the first, second, and third together have 27 denarii; the second, third, and fourth together have 31; the third, fourth, and first have 34; and the fourth, first and second have 37. How much does each have?"

Cardano's Solution of the Cubic Equation

Gerolamo Cardano (1501-1576) was the first person to publish a solution to the solution of the cubic equation $x^3 + cx = d$ in his *Ars magna, sive de regulis algebraicis* (*The Great Art, or On The Rules of Algebra*).

Cardano's Formula is

$$x = \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} + \frac{d}{2}} - \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} - \frac{d}{2}}$$

Nicolo Tartaglia (1499-1557) had first shown that if you select two numbers *u* and *v* such that

$u^3 - v^3 = d$ and $u^3v^3 = \left(\frac{c}{3}\right)^3$ then the solution to $x^3 + cx = d$ is $x = u - v$.

Exercise

Let's use Cardano's formula to solve the cubic $x^3 + 6x = 20$

EXAMPLE

Let's show that Tartaglia's conditions on *u* and *v* do indeed solve $x^3 + cx = d$

GroupWork

We should be able to derive Cardano's formula considering that Tartaglia's conditions correspond to solving a system of equations for two unknown quantities whose difference and product are known quantities.

First, we can show that $py^3 + qy^2 + ry = s$ can be made to look like $x^3 + cx = d$ by dividing by p, and using the transformation $y=x+\beta$ and selecting $\beta = -\frac{1}{3}\frac{q}{p}$

Second, solve the system $a\text{-}b = x$ and $ab = \frac{1}{3}c$ and re-derive Cardano's formula.

Notation issues

cosa	thing
censo	square
cubo	cube
radice	root
$\bar{p}$	più (plus)
$\bar{m}$	meno (minus)

Note, that Cardano did not use modern notation but would have written down the solution to the solution to $x^3 + 6x = 20$ as

$$\mathcal{R}\ \text{v}:\ \text{cub}\ \mathcal{R}\ 108\ \text{p}:\ 10\ \text{m}:\ \mathcal{R}\ \text{v}:\ \text{cub}\ \mathcal{R}\ 108\ \text{m}:\ 10$$

$$\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10}$$

Generally, this kind of algebraic manipulation is called **rhetorical** as opposed to **symbolic** which came later.

Don't Fear The Square Root!

Look at this problem by Antonio de Mazzinhi (1353-1383): "Find two numbers such that multiplying one by another makes 8 and the sum of their squares is 27."

Ans: $x = \dfrac{\sqrt{43}}{2}, y = \dfrac{11}{4}$

The solution involves choosing the first number is *un cosa meno la radice d'alchuna quantità* (a thing minus the root of some quantity) while the second number equals *una cosa più la radice d'alchuna quantità* (a thing plus the root of some quantity)

Exercise

Use Mazzinhi's method to solve the above problem.

The Beginning of Imaginary Numbers

Cardano also gave a formula for the solution of $x^3 = cx + d$, namely

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{c}{3}\right)^3}}$$

Rafael Bombelli (1526-1572) learned how to deal with examples of Cardano's formula for the cubic $x^3 = cx + d$ where the root becomes complex because $\left(\dfrac{d}{2}\right)^2 - \left(\dfrac{c}{3}\right)^3$ becomes negative. According to Katz, Bombelli proposed a name for such numbers as "neither positive (*più*) nor negative (*meno*)." What we call imaginary numbers, such as *bi* and *–bi*, Bombelli called *più di meno* (plus of minus) and *meno di meno* (minus of minus), respectively.

Bombelli gave multiplication rules for these new numbers, such as:

più di meno times *più di meno* equals *meno* and *meno di meno* times *più di meno* equals *più*

Practical Uses

Bombelli was able to show that the solution to $x^3 = 15x + 4$ is $x = 4$, even though by Cardano's formula one should get

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

However, if one assumes

$$\sqrt[3]{2 + \sqrt{-121}} = a + \sqrt{-b}$$

$$\sqrt[3]{2 - \sqrt{-121}} = a - \sqrt{-b}$$

One can obtain the equations $a^2+b=5$ and $a^3-3ab=2$ which Bombelli carefully showed has the solution $a=2$ and $b=1$.

Using this information, we can obtain the solution to the cubic to be $x=4$. Bombelli was able to use this knowledge to solve previously "unsolvable" quadratic equations like $x^2 + 20x = 4$

The Analytic Art

François Viète (1540-1603) developed theories for solving problems based on the work of the Greeks and published a work called *In artem analyticem isagoge* (*Introduction to the Analytic Art*) in 1591. Pappus had divided analysis into two parts: "problematic analysis" and "theorematic analysis."

Viète renamed these kinds of analysis and added a third.

Problematic analysis became **zetetic analysis** (the procedure by which one transforms a problem into an equation linking the unknown and various knowns).

Theorematic analysis became **poristic analysis** (the procedure exploring the truth of a theorem by appropriate symbolic manipulation)

Exegetics is the art of transforming an equation found by zetetic analysis to find a value for the unknown.

Viète is also known for his introduction of new symbols to represent terms, and was one of the first people to do algebra in a symbolic, not rhetorical manner.

Viète would write the equation $A^2 + 2BA = Z$ as "A quad + B2 in A equals Z plane" and its solution as

$A = \sqrt{Z^2 + B^2} - B$ becomes A is $l.\overline{Z\text{plane} + B\text{quad}} - B$ which Katz records as the first occurrence of the quadratic formula as we understand it today, in symbolic form.

Viète wrote Cardano's formula for the equation "A cube – B plane 3 in A equals Z solid 2" as

$$\text{A is } l.c.\overline{Z\text{solid} + l.\overline{Z\ \text{solidsolid} - B\ \text{planeplaneplane}}} + l.c.\overline{Z\text{solid} - l.\overline{Z\ \text{solidsolid} - B\ \text{planeplaneplane}}}$$