Worksheet 8

TITLE Early trig and algebra: the work of Ptolemy, Hipparchus, Diophantus and Hypatia
CURRENT READING: Katz §5-6 (pp 133-193) and Boyer & Merzbach, §8 (pp 142-174)

SUMMARY
The Greek Era of mathematics comes to and end but we can see the beginnings of trigonometry and algebra from the works of Ptolemy, Hipparchus, Diophantus and Hypatia.

NEXT: Medieval Mathematics
NEXT READING: Katz §10 (pp 324-363) and Boyer & Merzbach, §12-13 (pp 223-281)

Claudius Ptolemy of Alexandria(100-178 CE)
Ptolemy is well-known for his contributions to mathematics and astronomy. His great work the *Almagest* rivals Euclid's *Elements* for its equivalent impact on astronomy. It popularized the idea that one could use mathematics to describe the natural world, introducing the idea of a mathematical model.

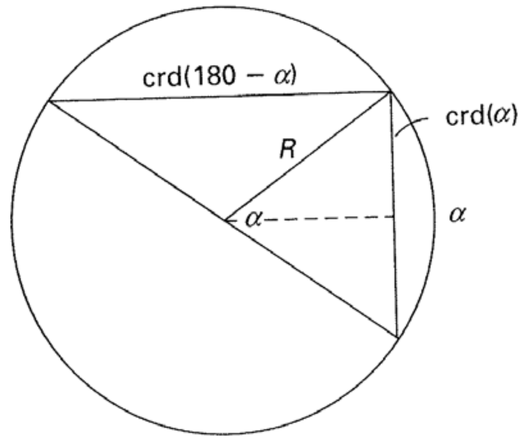
Pappus of Alexandria (fl. 300 CE)
Pappus was considered the last great Greek mathematician. By this time much of the amazing works of the Greeks had been lost and he wrote his *Collection* to try and restore that knowledge. He often took earlier results and generalized them to cover broader cases (like Pythagoras' Theorem).

Hypatia of Alexandria (c. 355-415 CE)
Hypatia was well-known as a commenter on the works of earlier Greek mathematicians such as Ptolemy, Archimedes, Apollonius, Diophantus and Pappus. She was also well-known as the daughter of Theon of Alexandria (fl 365 CE) who was responsible for an influential edition of the *Elements*. Her death at the hands of an angry mob who equated her mathematical knowledge and expertise with witchcraft is often the most widely known fact about her. However, some speculate that the Arabic versions of Diophantus' work are actually based on translations of her commentary, not the original text.

Diophantus of Alexandria (fl. 300 BCE)
Little is known about the life of Diophantus but he is notable for his great work *Arithmetica*, which he said was divided into 13 books. Sadly, only ten of the books are now known, with six having survived in the original Greek and four others from Arabic translations. His main claim to fame is the use of (abstract) symbolism in the expression of equations.

Hipparchus of Bythnia (190-120 BCE)
Hipparchus was one of the earliest people to create a table of chords (lengths) corresponding to arcs for a series of angles in the circle. He popularized the idea of dividing the circle into 360 parts, as he was regularizing results from the Babylonians (thus using base 60).

Hipparchus and the beginning of trigonometry



½ crd(α) /R =sin α/2

$$crd(\alpha) = 2R\sin\frac{\alpha}{2}$$

$$crd(180-\alpha) = \sqrt{(2R)^2 - (crd(\alpha))^2}$$

$$=2R\cos(\frac{\alpha}{2})$$

$$crd^2(\frac{\alpha}{2}) = R(2R - crd(180-\alpha))$$

Hipparchus of Bythnia (190-120 BCE) defined the length of a chord subtended by an angle α, denoted chord(α) or crd(α) by Katz. This marked the beginning of trigonometry as we know it today.

Hipparchus constructed a table of chords and used it to make astronomical calculation of surprising accuracy. He used a sexagesimal approximation of π to be 3;8,30 and assuming that there were 6,0,0 minutes (360 degrees divided into 60 minutes) in a circle he computed that a radius of a circle had to be 3438 minutes long, or 57,18 (in sexagesimal).

He calculated the length of the solar year to be 365 ¼ days, less 4 minutes, 48 seconds (off by 6 minutes from modern calculations) and the length of the lunar month to be 29 days, 12 hours, 44 minutes, 2 ½ seconds (less than 1 second off). Source: G. Donald Allen's *Ancient Greek Mathematics*.

Hipparchus' work was exceeded by the work of Claudius Ptolemy, who produced a table of chords from every angles from one-half a degree up to 180 degrees (in sexagesimal, of course). See Table 5.1 of Katz (given below)

Arcs	Chords	Sixtieths	Arcs	Chords	Sixtieths
$\frac{1}{2}$	0;31,25	0;1,2,50	6	6;16,49	0;1,2,44
1	1;2,50	0;1,2,50	47	47;51,0	0;0,57,34
$1\frac{1}{2}$	1;34,15	0;1,2,50	49	49;45,48	0;0,57,7
2	2;5,40	0;1,2,50	72	70;32,3	0;0,50,45
$2\frac{1}{2}$	2;37,4	0;1,2,48	80	77;8,5	0;0,48,3
3	3;8,28	0;1,2,48	108	97;4,56	0;0,36,50
4	4;11,16	0;1,2,47	120	103;55,23	0;0,31,18
$4\frac{1}{2}$	4;42,40	0;1,2,47	133	110;2,50	0;0,24,56

EXAMPLE

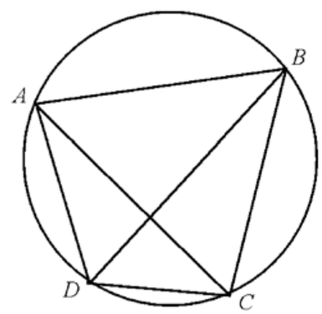
Let's try and confirm the value for crd(120º) since we know that crd(60º) = 57,18 which is equal to the radius *R* of the circle. (Why is crd(60º)=*R*?)

The Almagest
Ptolemy published *Mathematical Collection* (*Mathematiki Syntaxis*) which was translated into Arabic and because it was the predominant astronomical work for centuries. It became known as *megisti syntaxis* (the greatest collection) or "*al-magisti*" or in English, the *Almagest*.

Almost nothing is known about Ptolemy's personal life but he developed a mathematical model which described the motion of the sun, moon and known planets that was used until the 16th century.

Ptolemy's Theorem: *Given any quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.*

Theorem. $|AC|\,|BD| = |AD|\,|BC| + |AB|\,|DC|$

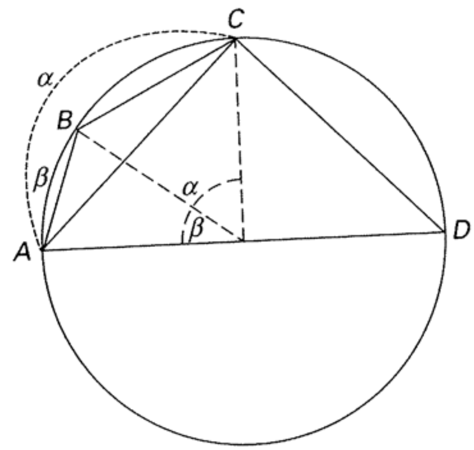


Ptolemy's Theorem $|AC||BD| = |AD||BC| + |AB||DC|$

EXAMPLE
Let's try and work through the proof of the theorem (Katz, page 147).

Applying Ptolemy's Theorem

We can reproduce some trigonometric identities. Consider the figure below (Katz, Figure 5.18):



It turns out that letting AD=crd(α) and AB=crd(β) then BC=crd(α- β).
Applying Ptolemy's Theorem to the quadrilateral ABCD produces:

$$\mathbf{120\ crd(\alpha-\beta) = crd(\alpha)\ crd(180-\beta) - crd(\beta)\ crd(180-\alpha)}$$

Which Katz claims can easily be shown to be equivalent to the well-known sine difference formula

$$\mathbf{\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)}$$

Heron's Formula(s)

Heron of Alexandria worked out a lot of formulas for the areas of plane figures, the most famous of which is

$$\textbf{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

Where s=$\frac{1}{2}(a+b+c)$ and the lengths of the three sides are a,b and c. Some have attributed this formula to Archimedes although it appears in Heron's *Metrica*.

Heron also gave formulas for A_n, the areas of regular polygons with n sides

$$A_3 \approx \frac{13}{30}a^2 \quad A_5 \approx \frac{5}{3}a^2 \quad A_7 \approx \frac{43}{12}a^2$$

He used A=$\frac{11}{14}d^2$ as the area of a circle of diameter d, making use of Archimedes approximation of $\frac{22}{7}$ for π.

Symbolism of Diophantus

He used various symbols to represent the unknown (what we call *x*)

$$x = \begin{cases} \zeta \quad S \quad \zeta' \\ y \quad\quad 'S^{\circ} \quad \alpha\rho \end{cases}$$

Unknown powers of *x* were denoted using Δ and/or K.

$$\begin{aligned} x^2 &:= \Delta^{\Upsilon} \\ x^3 &:= K^{\Upsilon} \\ x^4 &:= \Delta^{\Upsilon}\Delta \\ x^5 &:= \Delta K^{\Upsilon} \\ x^6 &:= K^{\Upsilon}K \end{aligned}$$

There was no equivalent symbol to + for addition. The default operation (no space between symbols) represented addition.

$$\pitchfork \;:=\; \text{minus} \qquad \overset{\circ}{M} := \text{units}$$

These symbols would be combined to represent what we know as a polynomial now.

EXAMPLE

$$K^{\Upsilon}\alpha\,\Delta^{\Upsilon}i\gamma\,\zeta\varepsilon\;\overset{\circ}{M}\,\beta = x^3 + 13x^3 + 5x + 2$$

What algebraic expression do these collections of symbols on the left represent?

$$K^{\Upsilon}\alpha\,\zeta\eta \pitchfork \Delta^{\Upsilon}\varepsilon\;\overset{\circ}{M}\,\alpha$$

$$\Delta^{\Upsilon}i\varepsilon \pitchfork \overset{\circ}{M}\,\lambda\theta$$

symbol	value	symbol	value	symbol	value
α	1	ι	10	ρ	100
β	2	κ	20	σ	200
γ	3	λ	30	τ	300
δ	4	μ	40	υ	400
ϵ	5	ν	50	ϕ	500
ϛ	6	ξ	60	χ	600
ζ	7	o	70	ψ	700
η	8	π	80	ω	800
θ	9	ϙ	90	ϡ	900

Exercise

Write $x^2 - 2x + 1$ using Diophantine notation.

Make up your own polynomial and trade with your nearest neighbor!

Modern:

Diophantine:

Diophantine equations

Diophantus only allowed solutions which were positive integers. Equations which satisfy this property are called **Diophantine equations**. These are still active areas of research today.

Diophantus generally solved two different kinds of equations in his most well-known work, called *Arithmetica*: "determinate equations" (single variable) and indeterminate equations (two or more unknowns in a single equations)

EXAMPLE

Problem II-8: ***To divide a given square number into two squares.***

Let $x^2 + y^2 = b^2$ where b^2 is "the given square."
Take any value for a and let $y=ax-b$.

Show that $x = \frac{2ab}{a^2+1}$ and $y = \frac{b(a^2-1)}{a^2+1}$ solves the problem.

GroupWork

Problem IV-31: ***To divide unity into two parts so that, if given numbers are added to them, respectively, the product of the two sums is a square.***

In modern symbols this problem can be represented algebraically as: $(x+a)(1-x+b) = y^2$

Note: that this problem has _____ variable(s) and _________ parameter(s).

Diophantus chose a=3 and b=5 and showed the solutions are then $\frac{6}{25}$ and $\frac{19}{25}$