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Worksheet 7

TITLE Archimedes and Apollonius
CURRENT READING:
[Archimedes] Katz, §4.1-4.3 (pp 94-112) and Boyer & Merzbach, §6 (pp 109-126)
[Apollonius] Boyer & Merzbach, §7 (pp 127-141) and Katz, §4.3-4.5 (pp 113-127).

SUMMARY
Archimedes (c. 287 BCE to 212 BCE) and Apollonius of Perga (c. 250 to 175 BCE), along with Euclid (fl. 300 BCE) comprise the “holy trinity” in the pantheon of Greek mathematicians.

NEXT: Ptolemy, Diphantus and Hypatia: Early trigonometry and early algebra
NEXT READING: Katz §5-6 (pp 133-193) and Boyer & Merzbach, §8 (pp 142-174)

Archimedes of Syracuse

How do we know what we know?

No surviving originals of the work of Archimedes exist. Most of the information we know about Archimedes life (and death) comes from a biography of the Roman soldier Marcellus written by Plutarch.

Many of Archimedes’ results were communicated in letters to people like Eratosthenes, who was chief librarian at Alexandra.

Archimedes wrote several books, which like many Greek works were copied and translated into the Arabic world.

The Archimedes *palimpsest* was discovered in Istanbul in 1899 and in 1906 translated by the great mathematical historian Heiberg. A palimpsest is a parchment in which the original text is washed off (but usually still partially visible) and a new text is overlaid on top the first. The reasons for this practice were 1) parchment was expensive and rare 2) it was considered virtuous by some religions to over-write “pagan” texts by religious texts.

Archimedes Anecdotes

“Eureka, Eureka!”

“Give me a place to stand on, and I can move the earth.”

The Death of Archimedes

The Grave of Archimedes

EXAMPLE

Proposition 3 of Archimedes' *Measurement of the Circle* is:

The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$.

How did Archimedes produce this accurate approximation of π?

He used Eudoxus' Method of Exhaustion along with a circumscribed polygon and an inscribed polygon to obtain ratios of the lengths of the sides of each (approximating the circle as the number of sides of the polygon increased) to the ratio of the radius of the circle.

He ended up with a recursive process which can work with a polygon of up to *n* sides which he pursued to *n*=96 to obtain his famous estimate.

The Sphere Inscribed By A Cylinder

The achievement Archimedes was most proud of was the result that the volume of a sphere is two-thirds of the volume of a circumscribed cylinder.

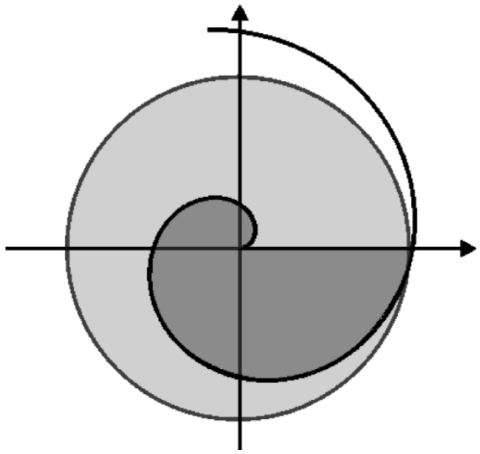
He also proved the result that the sphere also has two-thirds the surface area of the circumscribed cylinder.

Exercise

Confirm these result using the known formulas for the volume(surface area) of a cylinder and the volume(surface area) of a sphere. (Draw a picture representing the problem.)

GroupWork

The Archimedes spiral is the curve given by $r = a\theta$ (in polar co-ordinates). Archimedes showed that the area enclosed by one full revolution of the spiral is 1/3 of the area of the circle with center at the origin and radius equal to the spiral arc's distance from the origin after one full revolution.



Quadrature of the Parabola

Using the Method of Exhaustion, Archimedes showed that the area of a parabolic segment is 4/3 the area of a triangular segment having the same base and height. (Note: the slope of the parabola where the apex of the triangle touches the curve is parallel to the other side.)

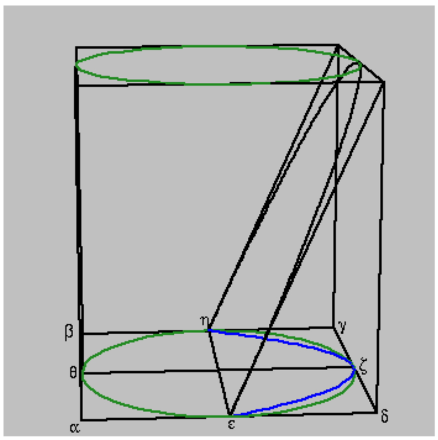
Let's show this result for various different triangles and parabolas, especially the generic $y = a - bx^2$

Solid Geometry

Proposition 14 in *The Method*: *If a cylinder is inscribed in a rectangular parallelepiped with square base, and if a plane is drawn through the center of the circle at the base of the cylinder and through one side of the square forming the top of the parallelepiped, then the segment of the cylinder cut off by this plane has a volume equal to one sixth of the entire parallelepiped*.

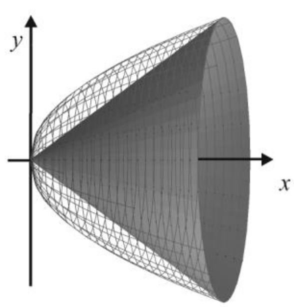
Exercise

Confirm that the volume of the parallelepiped is given by $2\int_0^r \int_0^{\sqrt{r^2-x^2}} z\,dy\,dx$ where $z = \frac{h}{r}x$ and h is the height of the parallelepiped and r is the radius of the cylinder.



Proposition 21 in *Of Conoids and Spheroids*: *Any segment of a paraboloid of revolution is half as large again as the cone or segment of a cone which has the same base and the same axes.*

Let's use Calculus to confirm this result while acknowledging how difficult it was for Archimedes to do it using the Method of Exhaustion.



Apollonius a.k.a. "The Great Geometer"
Appollonius coined the terms "parabola," "hyperbola" and "ellipse" in his seminal work *On Conics*.

What's amazing is the number of results that he was able to achieve without knowing about a coordinate system or algebra. It is all based on geometric reasoning.

The Cube Doubling Problem Is Solved
Recall that one of the (three) classic famous problems of antiquity is "Given a cube, how does one construct a cube of double the volume?" Basically, this is about constructing a length that is $\sqrt[3]{2}$ of another length.

Hippocrates had showed that one needs to obtain lengths which are in the ratio $a:x=x:y=y:2a$ where a is your original length and x is a length so that $(a:x)^3=1:2$.

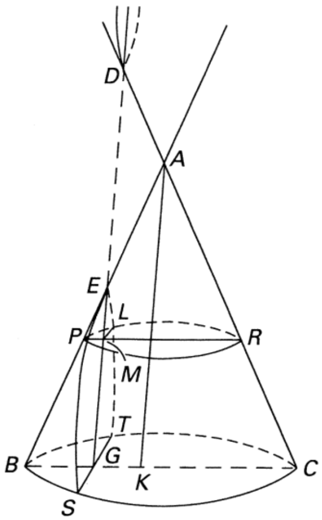
As Katz notes on page 112, algebraically, this is equivalent to solving simultaneous any two of the following equations $x^2 = ay$, $y^2 = 2ax$ or $2a^2 = xy$. Each of these curves happens to be a conic section (**name them**), so the cube doubling problem can be thought of as a curve intersection problem.

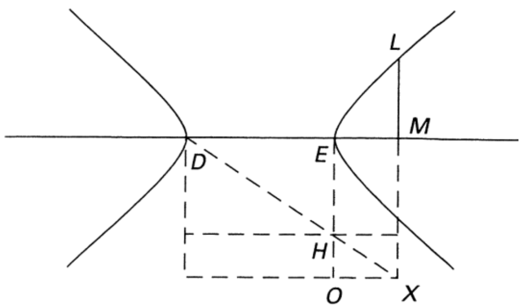
Exercise

Show that the solution of these simultaneous equations leads to $x=\sqrt[3]{2}\,a$

Apollonius' Definitions Of The Conics

Hyperbola (*yperboli* or "exceeding")





$$\frac{DE}{EH} = \frac{AK^2}{BK \cdot KC}$$

$$\frac{AK}{BK} = \frac{EG}{BG} = \frac{EM}{MP} \quad \text{and} \quad \frac{AK}{KC} = \frac{DG}{GC} = \frac{DM}{MR}.$$

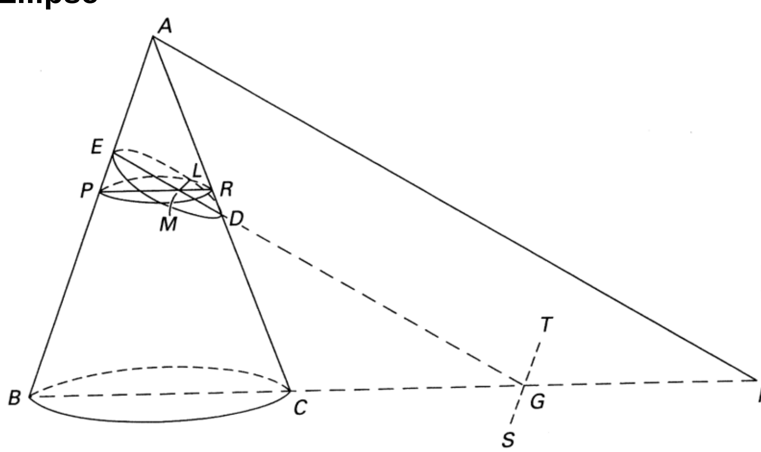
$$\frac{DE}{EH} = \frac{EM \cdot DM}{MP \cdot MR}.$$

$$\frac{DE}{EH} = \frac{DM}{MX} = \frac{DM}{EO} = \frac{EM \cdot DM}{EM \cdot EO}.$$

$MP \cdot MR = EM \cdot EO$ $\quad LM^2 = EM \cdot EO$

$EO = EH + HO$

Ellipse



$$\frac{DE}{EH} = \frac{DM}{MX} = \frac{DM}{EO} = \frac{EM \cdot DM}{EM \cdot EO}$$

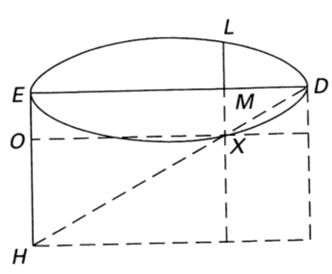
$MP \cdot MR = EM \cdot EO$

$LM^2 = EM \cdot EO$

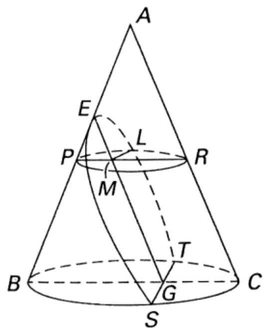
$EO = EH - HO$

(*ellipsis* or "deficient")

Let
LM=y
EM=x
EH=p
DE=2a



Parabola



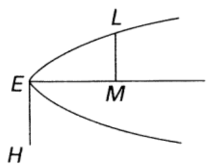
$$\frac{EH}{EA} = \frac{BC^2}{BA \cdot AC}.$$

$$\frac{EH}{EA} = \frac{MR \cdot PM}{EA \cdot EM}$$

$$\frac{EH}{EA} = \frac{EH \cdot EM}{EA \cdot EM}$$

$$MR \cdot PM = EH \cdot EM$$

$$LM^2 = EH \cdot EM$$



Let
LM=y
EM=x
EH=p

EXAMPLE

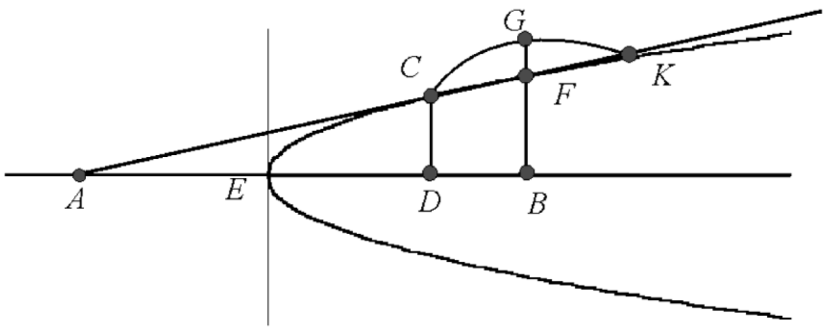
Let's derive the algebraic formulas for the ellipse, parabola and hyperbola from the geometric pictures above.

$y^2 = x\left(p + \frac{p}{2a}x\right)$ $y^2 = x\left(p - \frac{p}{2a}x\right)$ $y^2 = px$

GroupWork

Katz, Chapter 4, Problems #19-20. Let's try and prove the following results using Calculus (and modern coordinate systems)

Proposition I-33. If AC is constructed, where $|AE| = |ED|$, then AC is tangent to the parabola.



Proposition I-34. (ellipse) Choose A so that

$$\frac{|AH|}{|AG|} = \frac{|BH|}{|BG|}$$

Then AC is tangent to the ellipse at C.