Worksheet 6

TITLE Euclid's contribution to mathematics (in areas besides Geometry)
CURRENT READING: Katz, §3.6-3.7 (pp 77-82); Boyer & Merzbach, (pp 103-106)

SUMMARY
We will take a quick tour at some of the numerous contributions Euclid the "father of Geometry" made to other fields of mathematics.

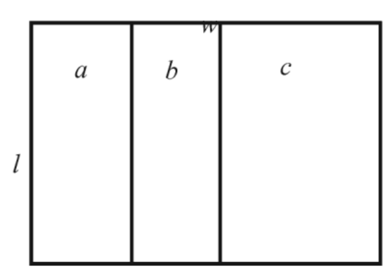
NEXT: Archimedes and Apollonius
NEXT READING:
[Archimedes] Katz, §4.1-4.3 (pp 94-112) and Boyer & Merzbach, §6 (pp 109-126)
[Apollonius] Boyer & Merzbach, §7 (pp 127-141) and Katz, §4.3-4.5 (pp 113-127).

Book II of *Elements*
The book is primarily involved with what is often called "geometric algebra" and is reminiscent of the work of the Babylonians. In fact, there is debate on how much of Book II is really just a compilation of previous work from Babylonian mathematicians.

Definition. *Any rectangle is said to be contained by the two straight lines forming the right angle.*

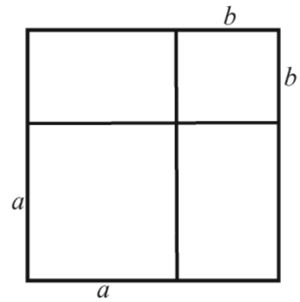
The point to notice here is that Euclid does not think of the two straight lines as two lengths that are multiplied together to produce an area. There is no process or concept of an arbitrary length being multiplied by another to produce an area.

Proposition II-1. *If there are two straight lines, and one of them is cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the uncut straight line and each of the segments.*



Exercise
What algebraic property does the figure represent?

Proposition II-4. *If a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments*.

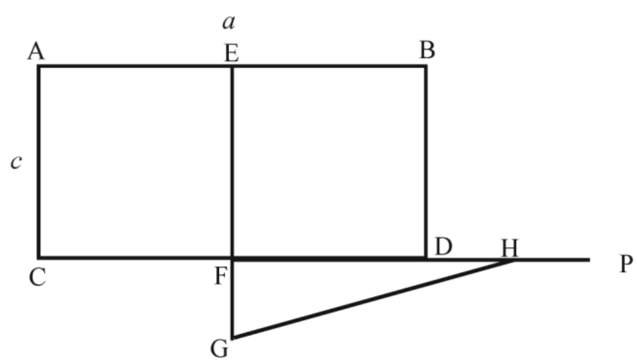


EXAMPLE

Can you see how this figure represent the binomial theorem for $n=2$?

Proposition II-14. *To construct a square equal to a given rectilinear figure*.

Algebraically, this corresponds to find a length *x* which solves $x^2 = ac$



Proof. Assume $a > c$. Solve $x^2 = ac$. Construct at the midpoint of AB, and produce the line EG of length $(a+c)/2$. Therefore length of the segment FG is $(a-c)/2$. Extend the line CD to P and construct the line GH of length $(a+c)/2$ (H is on this line.). By the Pythagorean theorem the length of the line FH has square given by

$$\left(\frac{a+c}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2 = ac$$

Book IV: Number Theory

The **Euclidean algorithm** for finding the greatest common divisor of two numbers.

Given two numbers *a* and *b* with $a>b$ subtract b from as many times as "possible."
If there is a remainder *c* (which must be less than *b*) then subtract *c* from *b* as many times as possible and continue until one ends with a number *m* which is the greatest common divisor. If this number is 1, then the two numbers *a* and *b* are said to be **relatively prime**.

The language that Euclid used was to say that *m* is the greatest common measure of *a* and *b*.

EXAMPLE

Use the Euclidean algorithm to find the greatest common divisor of 18 and 80. How is this different from the greatest common divisor of 9 and 40?

One of the classic results appearing in Euclid's number theory books is the result that there are an infinite number of prime numbers. Additionally, the **fundamental theorem of arithmetic** was well-known to Euclid: **Every number greater than 1 can be written as a *unique* product (up to the order of factoring) of prime numbers**.

Proposition VII-31. *Any composite number is measured by some prime number.*

Proposition VII-32. *Any number is either prime or is measured by some prime number.*

Proposition IX-20. *Prime numbers are more than any assigned multitude of prime numbers.*

PROOF

Let's look at Euclid's proof of IX-20 reproduced on page 80 of Katz. How would you improve it?

Book V: Ratio and Proportion
Theaetetus (417-369 BCE) and **Eudoxus** (408-355 BCE) were both Greek mathematicians that formalized the formulation of ratio and proportion. Eudoxus is well-known for his "Method of Exhaustion" and Theaetetus for his work on the regular polyhedra.

Euclid built on the work of these two and expanded it in Book V and a later book in the *Elements*.

Definition 5. *Magnitudes are said to be in the same ratio (or **proportional**), the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*

In modern notation, we would say the magnitudes a, b, c and d are *in the same ratio* $a:b = c:d$ if for all positive integers m and n, $ma > nc$ then $mb > nd$; and similarly for < and =.

This really means that when $ma>nc$ then $mb>nd$ and similarly for whenever $ma=nc$ this means that $mb=nd$ and whenever $ma<nc$ then $mb<nd$. In other words, the ratios a/c and b/d are always BOTH either less than, greater than or equal to the ratio n/m.

Definition

When $a:b = b:c$ this is said to mean that $a:c$ is the **duplicate ratio** of $a:b$.
In modern notation, we would say that the ratio of a to c is the square of the ratio a to b.
Continued proportion were sequences such that $a_1 : a_2 = a_2 : a_3 = a_3 : a_4 = \ldots$

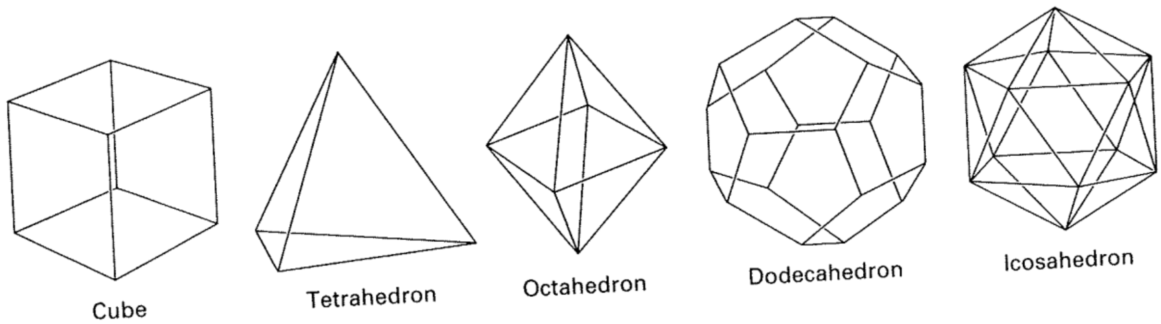
EXAMPLE

Proposition IX-35. *If as many numbers as we please are in continued proportion, and there is subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it*

Mathematically this is $(ar^n - a) : S_n = (ar - a) : a$
By re-arranging this expression what important formula (from Calculus 2) do you find?

The Regular Polyhedra
Also known as the Platonic Solids, these are objects where each face is a regular polygon.

In Book XII, Euclid proved that these were the only regular polyhedra and showed how they could be constructed and inscribed inside a sphere.



The Greeks were fascinated by the Platonic Solids and Plato assigned a natural Element to the four known at the time (4th century BCE):
cube=Earth, tetrahedron=Fire, octahedron=Air, icosahedron=Water.
Let's look at some other important characteristics of the Platonic solids.

	# of Faces	# of Vertices	# of Edges	# of Faces that meet at each Vertex	# of Sides of each Face
Tetrahedron					
Cube					
Octahedron					
Dodecahedron					
Icosahedron					

What patterns do you notice?

Which pairs of solids have the same number of edges? (These are known as the Dual Platonic Solids)

What's the relationship between the number of faces and the number of vertices for each of these pairs of solids?

What's the relationship between the number of faces that meet at each vertex and the number of sides of each face for these pairs of solids?