Worksheet 5

TITLE Introduction to Euclid and the *Elements*
CURRENT READING: Katz, §3.1-3.2 (pp 50-60); Boyer & Merzbach, (pp 93-101)

SUMMARY
We will be introduced to Euclid (*fl.* 300 BCE) and the most famous mathematical book of all time: Euclid's *Elements*.

NEXT: Euclid's contribution to mathematics (in areas besides Geometry)
NEXT READING: Katz, §3.6-3.7 (pp 77-82); Boyer & Merzbach, (pp 103-106)

Epistemology
How do we know what we know about Euclid and the *Elements*?

What do we know about Euclid?

What do we know about the *Elements*?

Online Resource for Euclid's *Elements*
You can access an online version of the *Elements* here:
https://mathcs.clarku.edu/~djoyce/java/elements/elements.html

Postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than to right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Axioms

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater that the part.

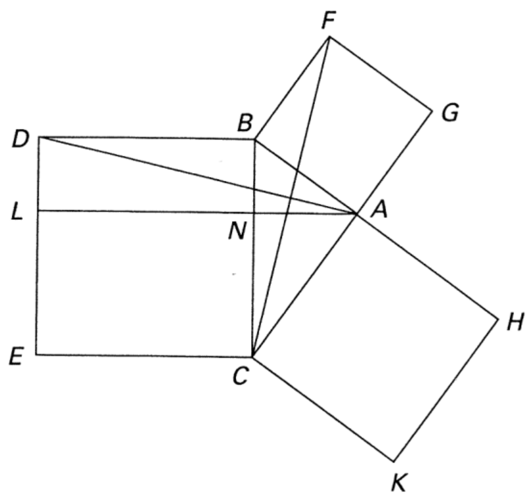
Discussion

Recall our definitions of **axioms** (statements that are taken to be true without argument by everyone) and **postulates** (statements that are taken to be true in the current context). What do you notice about Euclid's postulates and axioms?

Book I (of XIII) of the *Elements* is really all about getting to the point where Pythagoras' Theorem can be proved.

PROPOSITION I-47

In right-angled triangles the square on the hypotenuse is equal to the sum of the squares on the legs.



PROPOSITION I-46

On a given straight line to describe a square.

PROPOSITION I-4

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal sides also equal, then the two triangles are congruent.

PROPOSITION I-41

If a parallelogram has the same base with a triangle and is in the same parallels, the parallelogram is double the triangle.

PROPOSITION I-27

If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

same parallels, the parallelogram is double the triangle.

PROPOSITION I-16

In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

PROPOSITION I-34

In parallelograms the opposite sides and angles equal one another, and the diameter bisects the areas.

PROPOSITION I-29

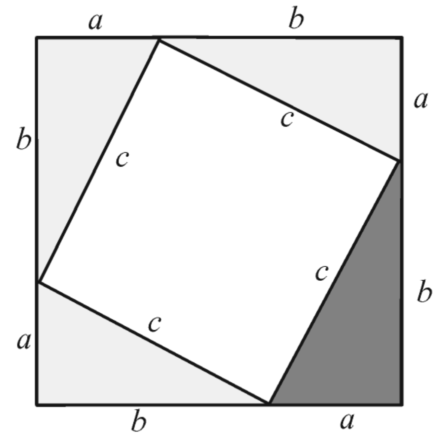
A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

GroupWork

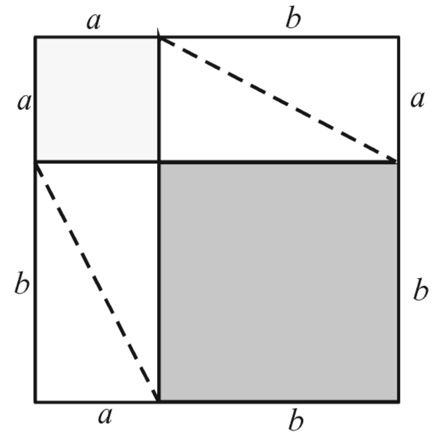
Let's repeat Katz's discussion (pp. 54-59) and work backwards through the propositions (theorems) that one needs in order to prove Pythagoras' Theorem.

Alternate Proof(s) of Pythagoras Theorem
G. Donald Allen claims there are over 300 published proofs of this theorem, and gives the following very pretty one

$$\begin{aligned}(a+b)^2 &= c^2 + 4(\frac{1}{2}\,ab)\\ a^2+2ab+b^2 &= c^2+2ab\\ a^2+b^2 &= c^2\end{aligned}$$

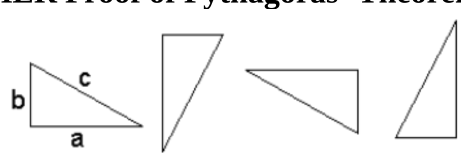


This proof is based upon Books I and II of Euclid's *Elements*, and is supposed to come from the figure to the right. Euclid allows the decomposition of the square into the two boxes and two rectangles. The rectangles are cut into the four triangles shown in the figure.

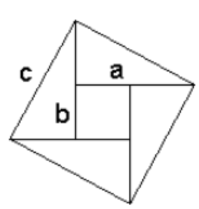


Then the triangle are reassembled into the first figure.

ANOTHER Proof of Pythagoras' Theorem



Now we start with four copies of the same triangle. Three of these have been rotated 90°, 180°, and 270°, respectively. Each has area **ab**/2. Let's put them together without additional rotations so that they form a square with side **c**.



The square has a square hole with the side (**a - b**). Summing up its area (**a - b**)² and 2**ab**, the area of the four triangles (4·**ab**/2), we get

$$\begin{aligned}\mathbf{c^2} &= \mathbf{(a-b)^2 + 2ab}\\ &= \mathbf{a^2 - 2ab + b^2 + 2ab}\\ &= \mathbf{a^2 + b^2}\end{aligned}$$