Senior Colloquium: *History of Mathematics*

Math 400 Fall 2019 — Library 355 T 3:05pm - 4:30pm

 — http://sites.oxy.edu/ron/math/400/19/

Worksheet 2: **Tuesday September 3**

TITLE Geometry and Algebra in Mesopotamia and Egypt
CURRENT READING Katz, 14-28; Boyer 12-28

SUMMARY
In today's class we will looks at some of the mathematics problems ancient Babylonians and Egyptians were able to solve as well as the techniques they used

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The Babylonians were quite skilled at geometry but approached the topic differently from modern mathematicians.

For example, they took the circumference to be the defining component of a circle (as opposed to the radius). For example they expressed the diameter as **0;20** (=1/3) and the area as **0;05** (=1/12). NOTE: these assume that the ratio of the circumference to the diameter of a circle that we know as π is exactly equal to 3 (instead of 3.1415926535789...)

For example, consider the figures below. Figure 1 consists of a shape formed by two circular arcs (comprised of a quarter of a circle) pasted together. It was known as "**the barge**." The shape in Figure 2 is similar, except it was formed by two circular arcs pasted together where each arc was one-third of a circle. It was called the "**bull's eye**."

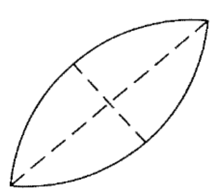
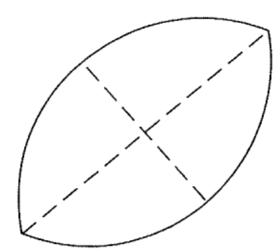
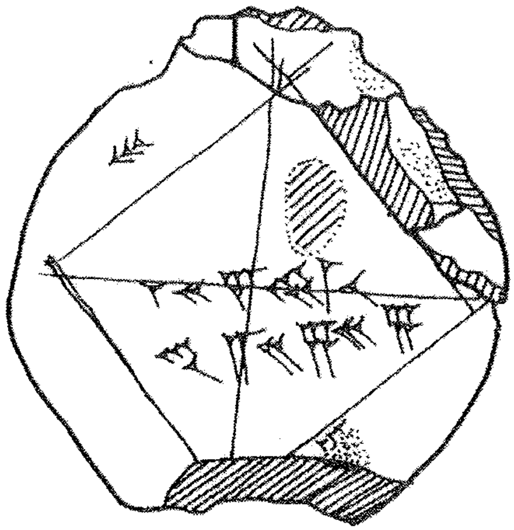
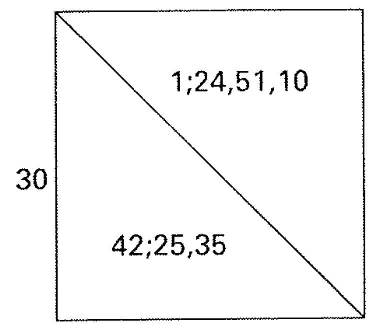
	

Fig 1: The "barge": two quarter-circle arcs	**Fig 2:** The "bull's-eye" two third-circle arcs
Area=$(2/9)a^2$ where a is the length of the arc	**Area=$(9/32)a^2$** where a is the length of the arc

GroupWork

Confirm one of the Babylonian formulas above (you will need the approximations $4\pi=12$ and $\sqrt{3}=7/4$)





Square Roots

Consider the figure above, which appears on a tablet known as YBC 7289. The tablet is from roughly 1800 BCE to 1600 BCE.

It is clear that 30 times 1;24,51,10 is exactly equal to 42;25;35 so this means that the Babylonians had an excellent approximation for the number $\sqrt{2}$ (1;24,51,10 is 1.414212963.)

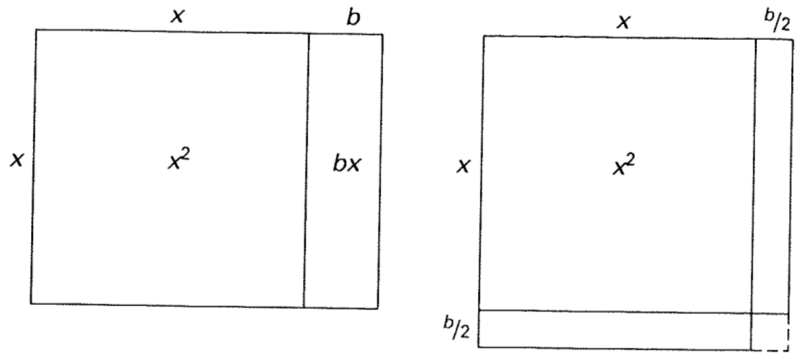
How did the Babylonians produce such an accurate approximation?
They used what is sometimes known as the Babylonian algorithm, which is also known as the "method of the mean."

Method of the mean. The method of the mean can easily be used to find the square root of any number. The idea is simple: to find the square root of 2, say, select x as a first approximation and take for another $2/x$. The product of the two numbers is of course 2, and moreover, one must be less than and the other greater than 2. Take the arithmetic average to get a value closer to $\sqrt{2}$. Precisely, we have

1. Take $a = a_1$ as an initial approximation.
2. Idea: If $a_1 < \sqrt{2}$ then $\frac{2}{a_1} > \sqrt{2}$.
3. So take
$$a_2 = (a_1 + \frac{2}{a_1})/2.$$
4. Repeat the process.

Note, that in modern mathematics we would consider this a recursive definition of a sequence $a_{n+1} = (a_n + \dfrac{2}{a_n})/2$. Can you prove the limit of this sequence is $\lim_{n\to\infty} a_n = \sqrt{2}$

Completing The Square and The Quadratic Formula



The Babylonians used a geometric technique for solving the quadratic equation $x^2 + bx = c$ which results in the quadratic formula $x = \sqrt{(b/2)^2 + c} - b/2$

EXAMPLE

Can one see how the above diagram represents the process of "completing the square" from algebra?

Solving linear systems. The solution of linear systems were solved in a particularly clever way, reducing a problem of two variables to one variable in a sort of elimination process, vaguely reminiscent of Gaussian elimination. **Solve**

$$\begin{aligned} \frac{2}{3}x - \frac{1}{2}y &= 500 \\ x + y &= 1800 \end{aligned}$$

Solution. Select $\tilde{x} = \tilde{y}$ such that

$$\tilde{x} + \tilde{y} = 2\tilde{x} = 1800$$

So, $\tilde{x} = 900$. Now make the model

$$x = \tilde{x} + d \qquad y = \tilde{y} - d$$

We get

$$\begin{aligned} \frac{2}{3}(900 + d) - \frac{1}{2}(900 - d) &= 500 \\ (\frac{2}{3} + \frac{1}{2})d + 1800/3 - 900/2 &= 500 \\ \frac{7}{6}d &= 500 - 150 \\ d &= \frac{6(350)}{7} \end{aligned}$$

So, $d = 300$ and thus

$$x = 1200 \qquad y = 600.$$

Egyptian Mathematical Methods
The Egyptians used a similar notion of proportionality to the one displayed above called the method of false position to solve linear equations.

For example, problems of the form: $x + ax = b$

The unknown x is called the **heep** or **heap**.

Problem 24 from the Ahmes Papyrus: **Find the heep if the heep and a seventh of the heep is 19.** (Solve $x + x/7 = 19$.)

Method of False Position (also see Katz, page 8)

Let g be the guess.
Substitute $g+ag = c$.
Now solve $c \cdot y = b$.
Answer: $x = g \cdot y$

EXAMPLE
Solve Problem 24 from the Ahmes papyrus using $g=7$.

The *Rhind* Papyrus and *Moscow Papyrus*
Two of the most famous mathematical artefacts in Egyptian mathematics are the *Rhind* payrus and the *Moscow* papyrus. Several of the homework problems and what we know about Egyptian mathematics comes from study of these two objects.

> The **Rhind Mathematical Papyrus** named for A.H. Rhind (1833-1863) from Scotland who purchased it at Luxor in 1858. Origin: 1650 BCE but it was written very much earlier. It is 18 feet long and 13 inches wide. It is also called the **Ahmes Papyrus** after the scribe that last copied it.
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> The **Moscow Mathematical Papyrus** purchased by V. S. Golenishchev (d. 1947) from Russia and later sold to the Moscow Museum of Fine Arts. Origin: 1700 BC. It is 15 ft long and 3 inches wide.

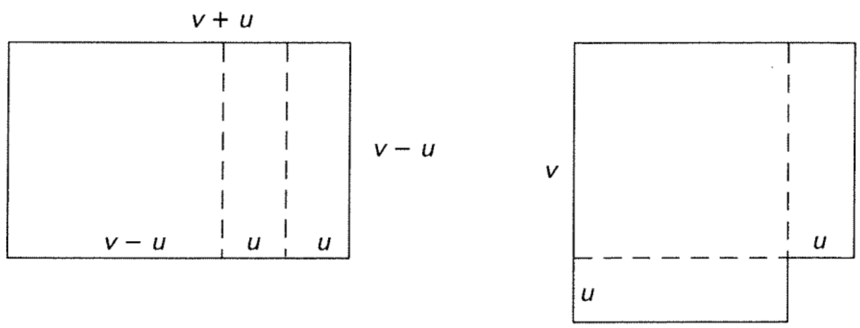
Plimpton 322 and Pythagorean Triplets

Plimpton 322 is a tablet discovered in 1945 dated from about 1700 BCE.

Numbers on the Babylonian tablet Plimpton 322, reproduced in modern decimal notation. (The column to the right, labeled y, does not appear on the tablet.)

$\left(\frac{d}{y}\right)^2$	x	d	#	y
1.9834028	119	169	1	120
1.9491586	3367	4825	2	3456
1.9188021	4601	6649	3	4800
1.8862479	12,709	18,541	4	13,500
1.8150077	65	97	5	72
1.7851929	319	481	6	360
1.7199837	2291	3541	7	2700
1.6845877	799	1249	8	960
1.6426694	481	769	9	600
1.5861226	4961	8161	10	6480
1.5625	45	75	11	60
1.4894168	1679	2929	12	2400
1.4500174	161	289	13	240
1.4302388	1771	3229	14	2700
1.3871605	28	53	15	45

The tablet is helping to find whole number solutions to $x^2 + y^2 = d^2$ by re-writing it as $\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{d}{y}\right)^2$ which if $u = \frac{x}{y}$ and $v = \frac{d}{y}$ becomes $u^2 + 1 = v^2$ which is equivalent to $(u+v)(u-v) = 1$. Then we can think of this expression using the figures below (which represent a rectangle of area 1 which has sides *v*-*u* and *v*+*u*) converted into an L-shaped object called a **gnomon**:



From Katz, page 21:

To calculate the entries on the tablet, it is possible that the author began with a value for what we have called $v + u$. Next, he found its reciprocal $v - u$ in a table and solved for $u = \frac{1}{2}[(v+u) - (v-u)]$. The first column in the table is then the value $1 + u^2$. He could then find v by taking the square root of $1 + u^2$. Since $(u, 1, v)$ satisfies the Pythagorean identity, the author could find a corresponding integral Pythagorean triple by multiplying each of these values by a suitable number y, one chosen to eliminate "fractional" values. For example, if $v + u = 2;15$ $(= 2\frac{1}{4})$, the reciprocal $v - u$ is 0;26,40 $(= 4/9)$. We then find $u = 0;54,10 = 65/72$. We would find v by taking half the sum of $v + u$ and $v - u$, but our scribe found v as $\sqrt{1 + u^2} = \sqrt{1;48,54,01,40} = 1;20,50$, or $\sqrt{1 + u^2} = \sqrt{1.8150077} = 1\frac{25}{72}$. Multiplying the values for u, v, and 1 by $1,12 = 72$ gives the values 65 and 97 for x and d, respectively, shown in line 5 of the table, as well as the value 72 for y. Conversely, the value of $v + u$ for line 1 of the table can be found by adding 169/120 (= 1;24,30) and 119/120 (= 0;59,30) to get 288/120 (= 2;24).

GroupWork

Pick another line in Plimpton 322 and confirm the entries in the tablet.