Topics in Applied Mathematics: *Mathematical Modeling*

Week 2: **Tuesday January 26**

TITLE The 5-Step Approach To Modeling; Introduction to Sensitivity and Robustness

CURRENT READING Meerschaert, Section 1.1, p. 3-19;

NEXT READING Prepare for guest lecture by Prof. Anthony Bonato on social networks by reading papers found at: `https://sites.oxy.edu/ron/math/396/21/models.html`

SUMMARY We will do an example of a mathematical modeling using the Meerschaert 5-Step Method and be introduced to the concepts of robustness and sensitivity of a model.

"The Pig Problem" is taken from *Mathematical Modeling* (4th Edition) by Mark M. Meerschaert. He popularizes the **5 Step Modeling Approach**:

1. Ask the question

   - Make a list of all the variables in the problem, including appropriate units.

   - Be careful not to confuse variables and constants.

   - State any assumptions you are making about these variables, including equations and inequalities.

   - Check units to make sure that your assumptions make sense.

   - State the objective of the problem in precise mathematical terms.

2. Select the modeling approach

   - Choose a general solution procedure to be followed in solving this problem.

   - Generally speaking, success in this step requires experience, skill, and familiarity with the relevant literature.

   - (In this class) we will usually specify the modeling approach to be used.

3. Formulate the model

   - Restate the question posed in step 1 in the terms of the modeling approach specified in step 2.

   - You may need to relabel some of the variables specified in step 1 in order to agree with the notation used in step 2.

   - Note any additional assumptions made in order to fit the problem described in step 1 into the mathematical structure specified in step 2.

4. Solve the model

   - Apply the general solution procedure specified in step 2 to the specific problem formulated in step 3.

   - Be careful in your mathematics. Check your work for math errors. Does your answer make sense?

   - Use appropriate technology. Computer algebra systems, graphics, and numerical software will increase the range of problems within your grasp, and they also help reduce math errors.

5. Answer the question

   - Rephrase the results of step 4 in nontechnical terms.

   - Avoid mathematical symbols and jargon.

   - Anyone who can understand the statement of the question as it was presented to you should be able to understand your answer.

EXAMPLE

Let's use the 5-Step Method to solve **the pig problem**.

A pig weighing 200 pounds gains 5 pounds per day and costs 45 cents a day to keep. The market price for pigs is 65 cents per pound but is falling 1 cent per day. When should the pig be sold?

Sensitivity Analysis

Suppose that one is uncertain about the exact value of a parameter in the problem. One can conduct what is known as a sensitivity analysis into how the model would change with respect to changes in parameter values.

DEFINITION: **Parameter**

A **parameter** is a known value in an equation, expression or system that represents a particular circumstance; when the parameter changes value the circumstances also change.

DEFINITION: **Sensitivity**

Suppose a parameter r results in an answer x to a problem. The sensitivity of x with respect to r is denoted as $S(x, r)$ and is calculated as

$$S(x, r) = \lim_{\Delta r \to 0} \frac{\Delta x / x}{\Delta r / r} = \lim_{\Delta r \to 0} \frac{\Delta x}{\Delta r} \frac{r}{x} = \frac{dx}{dr} \cdot \frac{r}{x}$$

DEFINITION: **robust** or **fragile** mathematical model

If the conclusions of a mathematical model do not vary greatly on how well the assumptions are satisfied then such a model is described as **robust**. If the conclusions of the model do vary greatly when its assumptions change it is known as **fragile**. Parameter sensitivity can be used as a quantitative measure of the robustness or fragility of a model.

GROUPWORK

Question | **What's the difference between a parameter and a variable?**

Question | **What are the parameters and variables in the Pig Problem?**

$\boxed{\text{EXAMPLE}}$

In the Pig Problem, what is the sensitivity of the time to sell x with respect to the rate r at which the price for pigs is falling?

$\boxed{\textbf{Exercise}}$

In the Pig Problem, what is the sensitivity of the time to sell x with respect to the rate g at which the weight of the pig is increasing per day?

The Pig Problem, Revisited

Reconsider the pig problem, but now take into account the fact that the growth rate of the pig is not constant. Assume that the pig is young, so that the growth rate is increasing, but in proportional to the pig's weight. When should the pig be sold for maximum profit?

(Your objective function for profit $f(x)$ versus time to sell x should be
$f(x) = (0.65 - 0.01x)(200e^{0.025x}) - 0.45x$.)