Special Topics in Advanced Math: *History of Mathematics*

Math 395 Fall 2023 | Fowler 310 TR 1:30pm - 2:55pm

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Class 21: **Tuesday November 14**

TITLE Other 19th Century Mathematicians: Galois, Riemann, Boole, Cantor, Poincaré, etc
READING: Katz, pp. 726-759; Boyer & Mertzbach, pp. 526-548

SUMMARY
We will discuss other mathematical happenings in the 19th century!

NEXT: Twentieth Century Mathematicians: Godel, Russell, Noether, Turing, Hilbert, etc
NEXT READING: Katz, pp. 874-903; Boyer & Mertzbach, pp. 548-586

Other Nineteenth Century Mathematicians

The great mathematical results in the Nineteenth Century we should be aware of are (apart from the work of Gauss), the debate over the definition of mathematical rigor (which we looked at last time by analyzing the actual words of Abel, Bolzano, Cauchy and Dedekind) and the re-examination of the foundations of mathematics that led to the development of set theory by Georg Cantor. Today we will be introduced to the names and accomplishments of just a sample of other mathematicians who lived and produced in the nineteenth century.

George Boole (1815-1864)
George Boole was the first professor of mathematics at Queen's College in Ireland and is most well-known for his contributions to the mathematization of logic in two works, *The Mathematical Analysis of Logic* (1847) and *Investigation of the Laws of Thought* (1854) where he introduced and summarized what is now known as Boolean algebra. Boolean algebra introduced the concept of treating logical operations as algebraic operations. Boole also contributed to differential equations, and in his 1859 book *Treatise on Differential Equations* made important linkages between the properties of differential operators and the rules of algebra.

Georg Cantor (1845-1918)
Cantor is most well known as the creator of set theory and the first person to posit the existence of multiple infinities (which he called transfinite numbers). In addition to proving astonishing results about infinite sets, Cantor provided a robust definition of the real numbers and invented the method of using one-to-one correspondences to compare infinite sets. Cantor was German and studied under Weierstrass in Berlin and eventually became a professor at Halle. He was later prevented from becoming a professor in Berlin due to opposition from Kronecker, who (along with Poincaré and Wittgenstein) despised Cantor's theories of infinite sets. Cantor suffered from mental illness for extended periods during the latter half of his life but lived long enough to see his mathematical results embraced by the mathematics community.

Johann Peter Gustav Lejeune Dirichlet (1805-1859)
Born in Prussia to parents from Belgium who were not wealthy, Lejeune Dirichlet (more commonly known as Dirichlet) is most well-known for providing the first formal definition of a function which is nearly identical to the modern one ("to any x there corresponds a single

finite *y*"). Dirichlet was a close associate of Gauss and was appointed to replace him at Gottingen when the Prince of Mathematics died in 1855. Dirichlet made several contributions in number theory, often building on or refining results of other mathematicians, especially Gauss. He is also known for the creator of the Dirichlet function, defined as "when x is rational, let $y=c$, and when x is irrational, let $y=d$, $d \neq c$" which is an example of a function which is not Riemann-integrable and is "nowhere continuous." Dirichlet built on the work of Fourier, Cauchy and Abel to prove conditions on when the Fourier series representation of a function $f(x)$ will converge to the function at all points where $f(x)$ is continuous.

Jean-Baptiste Joseph Fourier (1768-1830)
Fourier is most well known for his assertion (strongly resisted by Lagrange and Laplace) that any function could be represented by an infinite series of sines or cosines in his seminal work *Théorie analytique de la chaleur* (Analytical Theory of Heat). He is credited for the discovery of the greenhouse effect (i.e. that carbon dioxide in Earth’s atmosphere will lead to increased warming). He was one of the most prominent scientists in French society during his short life, serving as professor at the *École Polytechnique*, as Permanent Secretary of the French Academy of Sciences and as a political appointee of Napoleon to high office.

Evariste Galois (1811-1832)
The story of the short, tragic life of the brilliant Frenchman who died at the age of 20 after repeatedly failing to get into his country’s top academic institution, *École Polytechnique* despite publishing his first mathematics paper at the age of 17 (only two years after beginning to study the subject!) is an example of “truth stranger than fiction.” Galois submitted his work on the theory of equations to the French Academy of Sciences multiple times but it wasn’t published in his lifetime due to a series of unfortunate events, including opposition by Poisson and Cauchy and the untimely death of Fourier. The day before his fatal duel he stayed up all night summarizing his mathematical thoughts which he wrote down in a letter to a friend. The ideas in these letters were later published by Liouville and resulted in Galois now being considered one of the founders of the theory of equations and modern algebra, and has an entire branch of algebra known after him called Galois theory.

Carl Gustav Jacob Jacobi (1804-1851) was a Prussian mathematician famous for his contributions to the study of elliptic functions. Jacobi along with Abel and Legendre are generally given credit for the discovery of this important branch of mathematics. He was able to apply elliptic functions to prove both Fermat’s two-square theorem and Lagrange’s four-square theorem. Of course, he is also known for the “Jacobian,” the square matrix (and its determinant) consisting of all the partial derivatives of a multivariable vector function which has an equal number of dependent variables and components.

Leopold Kronecker (1823-1891) was born in a section of Prussia which later became Poland. He is well-known for the following quote "*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk"* ("God made the integers, all else is the work of man") which reflected his mathematical theory that all mathematical concepts should be defined in terms of the natural numbers or involve a finite number of steps. Kronecker made important contributions to number theory, the theory of equations, and elliptic function theory, among others. He had an outsized influence on nineteenth century mathematics as the editor of Leopold Crelle’s *Journal*, which was the most prestigious mathematics research journal for decades, and as co-convener (with Weierstrass) of the University of Berlin’s mathematics research seminar.

Henri Poincaré (1854-1912) was a Frenchman who made significant contributions to mathematics and physics. He was known for his fundamental contributions to topology, dynamical systems and mathematical physics. He was adamantly opposed to Cantor's approach to infinity and decried the concept of transfinite numbers. Early in the Twentieth Century he made what is known as the Poincaré conjecture (which was one of the most famous unsolved problems in Mathematics until a proof of the result was published by **Grigori Perelman** in 2006) and was elected as President of the French Academy of Sciences in 1905.

Karl Weierstrass (1815-1897) was a German mathematician who has been called "the father of analysis" and "the greatest mathematics teacher of the nineteenth century." He attended the University of Bonn in law and economics but became fascinated with mathematics and ended up leaving college without a degree but he was able to get certified as a mathematics teacher. After publication of two famous papers in Crelle's *Journal* he was offered an honorary degree from the University of Konigsberg and became a professor at the University of Berlin, where his lectures became legendary and he began its world-famous mathematical seminar in 1861. He is most well-known for the rigor and precision of his mathematical writing, and he is given credit for the modern definitions of "continuity," "uniform continuity" as well as his discovery/creation of the Weierstrass function, which is nowhere differentiable but uniformly continuous.

The Seven Daughters of Mathesis

Eves (576) discusses the role of women in mathematics and identifies seven women of note:

The seven mythical daughters of Atlas have become enshrined in the northern sky as the seven principal stars in the Pleiades cluster. In imitation, the seven mathematicians (Hypatia, Maria Gaetana Agnesi, Sophie Germain, Mary Fairfax Somerville, Sonja Kovalevsky, Grace Chisholm Young, and Amalie Emmy Noether) have become known as The Mathematical Pleiades, or The Seven Daughters of Mathesis. Not only were these women competent mathematicians, but they have inspired and enabled other women to enter mathematics. The sex barrier in mathematics of the nineteenth and early twentieth centuries was broken, and universities became open to the attendance and academic recognition of women and to the acceptance of women on their faculties.

There was formed in America in 1971 the Association for Women in Mathematics (open to both female and male members), founded to put male and female mathematicians on an equal footing. The male population has no inherent superiority in mathematical thinking or creativity, and there is today a rapid increase in the number of women among the topflight practitioners and creators of mathematics.

Discussion

Why do you think women have been and still are underrepresented in mathematics?

Other Notable 19th Century Mathematical Results

Below are some mathematical results associated with some of these 19th century mathematicians (i.e. other than Gauss and Cauchy)

Fermat's 2-Square Theorem

An odd prime p can be written as the sum of the square of two integers x and y if and only if $p \equiv 1 \pmod{4}$.

Lagrange's 4-Square Theorem (Bachet's Conjecture)

Every natural number can be represented as the sum of *up to* four integer squares:

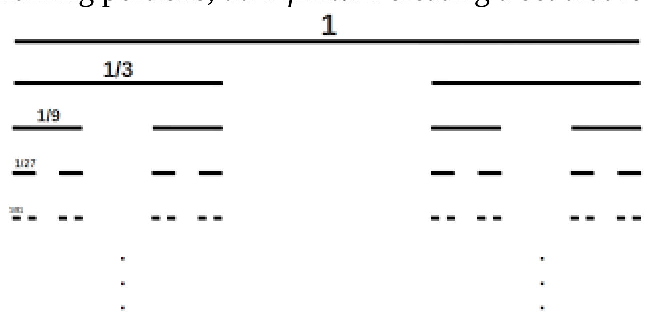
$$p = a^2 + b^2 + c^2 + d^2$$

Poincaré Conjecture

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

Cantor Set

The Cantor set (often called the Cantor ternary set) is the set of points on the real numberline between 0 and 1 where the middle third of the set is iteratively removed and then the process is repeated for the remaining portions, *ad infinitum* creating a set that looks like:



Elliptic Functions

The Jacobi elliptic functions are analogous to trigonometric functions but they are defined relative to an elliptic integral:

$$u = F(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

Where the elliptic sine function **sn(u)** and elliptic cosine function **cn(u)** are defined as $\mathrm{sn}(u)=\sin(\phi)$ and $\mathrm{cn}(u)=\cos(\phi)$.

Continuity versus Uniform Continuity (and convergence versus uniform convergence)

Weierstrass was instrumental in defining the concept of continuity and distinguishing it from uniform continuity.

Recall that continuity of a function is a local property, while uniform continuity is a global property of a function.

Weierstrass defined the concept of uniform convergence of sequence of functions to separate it from pointwise convergence during his analysis of power series and Fourier series.

Continuum Hypothesis

The continuum hypothesis is the conjecture that there is no set which has a cardinality that is greater than that of the natural numbers and less than the cardinality of the real numbers.

The continuum hypothesis is that the cardinality of the continuum (i.e. the set of points in a finite line segment) which is sometimes denoted **c** is exactly equal to $\aleph_1$. This idea is somewhat controversial in some mathematical circles. There are some that believe that (the continuum hypothesis is FALSE) and there are many transfinite numbers between $\aleph_1$ and **c** and there are some that believe (the continuum hypothesis is TRUE) so these two numbers are the same.

$$|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} = \aleph_1$$

The cardinality of the power set of the natural numbers is called $\aleph_1$.

THEOREM $\mathbf{c} > \aleph_0$

The cardinality of the real numbers is greater than the cardinality of the natural numbers.

DEFINITION: **One-to-One Correspondence**

A **one-to-one correspondence** is a function that operates between two sets where every element of one set is paired with exactly one element of the other set and every element of the other set is paired with exactly one element of the first set. This kind of function is also called a **bijection** or **bijective function**.

The **empty set** is the set which contains no elements. It is often denoted **{}** or $\emptyset$.

DEFINITION: **Power Set**

The **power set** of a set is the set of *all possible* subsets of a set. Note, the empty set is a sub- set of every set. If the given set is denoted A, then P(A) is the notation for the power set of A.

DEFINITION: **Cardinality**

The **cardinality** of a (finite) set is the number of elements in that set. The cardinality of

the empty set is zero. The notation for cardinality of a set S is $|S|$. So, $|\{\}| = 0$.

DEFINITION: **Uncountable**

An infinite set is said to be **uncountable** (or **nondenumerable**) if it has a cardinality that is greater than $\aleph_0$, i.e. it has more elements than the set of natural numbers.

DEFINITION: **Aleph Null** or $\aleph_0$

ℵ

The cardinality of the set of the natural numbers is denoted $\aleph_0$ and said to be "aleph null" or "aleph zero." Sets with this cardinality are often said to be **countable** or **countably infinite** or **denumerable**. $\aleph_0$ is said to be the first of the **transfinite numbers**.

GroupWork

Consider the following dozen infinite sets comparisons. Write down >, < or = between each pair of sets to indicate their relative size.

	SET #1	SET #2
1:	$\{1, 2, 3, 4, \ldots\}$ [all natural numbers]	[The natural numbers starting with 3] $\{3, 4, 5, 6, \ldots\}$
2:	$\{1, 2, 3, 4, \ldots\}$ [all natural numbers]	[All even natural numbers] $\{2, 4, 6, 8, \ldots\}$
3:	$\{1, 2, 3, 4, \ldots\}$ [all natural numbers]	[All odd natural numbers] $\{1, 3, 5, 7, \ldots\}$
4:	$\{1, 2, 3, 4, \ldots\}$ [all natural numbers]	[All unit fractions] $\{1, 1/2, 1/3, 1/4, \ldots\}$
5:	$\{1, 2, 3, 4, \ldots\}$ [all natural numbers]	[All points on an infinite line]
6:	[All points on a finite line segment]	[All points on an infinite line]
7:	$\{1, 3, 5, 7, \ldots\}$ [all odd natural numbers]	[All multiples of four] $\{4, 8, 12, 16 \ldots\}$
8:	$\{10, 20, 30, 40, \ldots\}$ [all multiples of 10]	[All multiples of four] $\{4, 8, 12, 16 \ldots\}$
9:	[All points on an infinite line]	[All points on a line 1" long]
10:	[All points inside a unit circle]	[All points on the circumference of a unit circle]
11:	[All points inside a unit circle]	[All points inside a unit square]
12:	[All points on a line 1/2" long]	[All points on a line 1" long]