Class 20: Thursday November 9

TITLE Abel, Bolzano, Cauchy and Dedekind: The Rigor Debate in the Nineteenth Century
READING: Katz, 766-787; Boyer & Mertzbach, 526-538; Eves, 563-576

SUMMARY
We will discuss the debate over the meaning of "rigor" that raged in the 19^{th} century.

NEXT: Other 19th Century Mathematicians: Galois, Riemann, Boole, Cantor, Poincaré etc
NEXT READING: Katz, pp. 766-787; Boyer & Mertzbach, pp. 452-460; Eves, 606-636

Niels Henrik Abel (1802-1829)
Abel's life was brief but he made an indelible mark upon the history of mathematics. At age 16 he wrote a paper proving the exponent in the binomial expansion was valid for all real numbers. He was also the first person to prove that polynomials of degree 5 do not have closed form solutions in terms of radicals (unlike quadratic equations, cubic equations and quartic equations). Several papers were published after his death (at the age of 26 due to tuberculosis in poverty!) that led to the development of group theory. The most prestigious annual prize in Mathematics is named after him. Abel Prizes have only been awarded (by the King of Norway) since 2002 but the notion of a prize on par with the Nobel Prizes was suggested in the lead up to the 100^{th} anniversary of Abel's birth when it was discovered there would be no Nobel Prize in Mathematics.

Bernard Bolzano (1781-1848)
Bolzano was a Czech mathematician who was familiar with the work of both Lagrange and Cauchy. Bolzano is most well-known for his definition of continuity that is very similar to Cauchy's. He was also known for his examples of function that refuted his contemporaries'' mathematical intuition at the time. Known as "monster functions," these include (i) a function that is not continuous anywhere; (ii) a function that is continuous at exactly one point; and (iii) a function that is continuous everywhere but nowhere differentiable.

Augustin-Louis Cauchy (1789-1857)
Cauchy was one of the most prolific mathematicians of all-time, probably second behind Euler.nCauchy was born in 1789 and initially educated by his father, although he began attending *Ecole Polytechnique* in 1805, where he gained attention from Lagrange and Laplace after placing second out of nearly 300 in the entrance exam. In 1807, Cauchy went on to attend *École des Ponts et Chaussées* (Schools of Bridges and Highways) to become a civil engineer. However, his engineering career was short-lived and by 1816 Cauchy was a full professor of mathematics at *École Polytechnique.*

Richard Dedekind (1831-1916)
Dedekind is most well-known for his contributions to analysis (i.e. the Dedekind cut can be used to formally define the real numbers) as well as to group theory. Dedekind was Gauss' was the first person to formally define a field and an ideal; he made important contributions to the debate over rigor in mathematics in the mid-nineteenth century.

The Nineteenth Century Debate Over Mathematical Rigor: The "Arithmetization of Analysis"

The Nineteenth Century is where we can trace a lot of the way that we do mathematics now got its start. Mathematicians like Abel, Bolzano, Cauchy, Dedekind, Kronecker, Weierstrass among others.

Katz compiles some definitions of these terms throughout time in sidebars on page 768, 770 and 783.

Definitions of "limit"

Leibniz (1684): If any continuous transition is proposed terminating in a certain limit, then it is possible to form a general reasoning, which covers also the final limit.

Newton (1687): The ultimate ratio of evanescent quantities . . . [are] limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*.

Maclaurin (1742): The ratio of $2x + o$ to a continually decreases while o decreases and is always greater than the ratio of $2x$ to a while o is any real increment, but it is manifest that it continually approaches to the ratio of $2x$ to a as its limit.

D'Alembert (1754): This ratio $[a : 2y + z]$ is always smaller than $a : 2y$, but the smaller z is, the greater the ratio will be and, since one may choose z as small as one pleases, the ratio $a : 2y + z$ can be brought as close to the ratio $a : 2y$ as we like. Consequently, $a : 2y$ is the limit of the ratio $a : 2y + z$.

Lacroix (1806): The limit of the ratio $(u_1 - u)/h$ . . . is the value towards which this ratio tends in proportion as the quantity h diminishes, and to which it may approach as near as we choose to make it.

Cauchy (1821): If the successive values attributed to the same variable approach indefinitely a fixed value, such that they finally differ from it by as little as one wishes, this latter is called the limit of all the others.

Definitions of "continuity"

Euler (1748): A continuous curve is one such that its nature can be expressed by a single function of x. If a curve is of such a nature that for its various parts . . . different functions of x are required for its expression, . . . , then we call such a curve discontinuous.

Bolzano (1817): A function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits . . . if [when] x is some such value, the difference $f(x + \omega) - f(x)$ can be made smaller than any given quantity provided ω can be taken as small as we please.

Cauchy (1821): The function $f(x)$ will be, between two assigned values of the variable x, a continuous function of this variable if for each value of x between these limits, the [absolute] value of the difference $f(x + \alpha) - f(x)$ decreases indefinitely with α.

Dirichlet (1837): One thinks of a and b as two fixed values and of x as a variable quantity that can progressively take all values lying between a and b. Now if to every x there corresponds a single, finite y in such a way that, as x continuously passes through the interval from a to b, $y = f(x)$ also gradually changes, then y is called a continuous function of x in this interval.

Heine (1872): A function $f(x)$ is continuous at the particular value $x = X$ if for every given quantity ϵ, however small, there exists a positive number η_0 with the property that for no positive quantity η which is smaller than η_0 does the absolute value of $f(X \pm \eta) - f(X)$ exceed ϵ. A function $f(x)$ is continuous from $x = a$ to $x = b$ if for every single value $x = X$ between $x = a$ and $x = b$, including $x = a$ and $x = b$, it is continuous.

Definitions of "function"

Johann Bernoulli (1718): I call a function of a variable magnitude a quantity composed in any manner whatsoever from this variable magnitude and from constants.

Euler (1748): A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.

Euler (1755): When quantities depend on others in such a way that [the former] undergo changes themselves when [the latter] change, then [the former] are called functions of [the latter]; this is a very comprehensive idea which includes in itself all the ways in which one quantity can be determined by others.

Lacroix (1810): Every quantity whose value depends on one or more other quantities is called a function of these latter, whether one knows or is ignorant of what operations it is necessary to use to arrive from the latter to the first.

Fourier (1822): In general, the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there is an equal number of ordinates $f(x)$. All have actual numerical values, either positive or negative or null. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity.

Heine (1872): A single-valued function of a variable x is an expression which for every single rational or irrational value of x is uniquely defined.

Dedekind (1888): A function ϕ on a set S is a law according to which to every determinate element s of S there belongs a determinate thing which is called the transform of s and denoted by $\phi(s)$.

GroupWork

Give the formal mathematical modern definition of **limit, continuity** and **function**. How does the modern definition compare to the historical ones presented earlier?
Draw pictures to illustrate your definitions!

Cauchy's definition of the integral
Suppose *f* (x) is continuous on a closed interval [x_0,X] and there are *n-1* values in this interval such that $x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = X$ then the sum S given by

$$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \ldots + (X - x_{n-1})f(x_{n-1})$$

Which can also be re-written

$$\sum_{k=1}^{n}(x_k - x_{k-1})f(x_{k-1})$$

Cauchy showed that the partition (the choice of values in the interval) were irrelevant to the final value of *S* as long as *n* gets larger and larger, and he called the limit of this process the definite integral which is written

$$\int_{x_0}^{X} f(x)dx$$

Mean Value Theorem for Integrals
For some $0 \leq \theta < 1$

$$\int_{x_0}^{X} f(x)dx = (X - x_0)f[x_0 + \theta(X - x_0)]$$

Fundamental Theorem of Calculus

Given *f(x)* is continuous in [x_0,X] and x is in [x_0,X] and $F(X) = \int_{x_0}^{X} f(x)dx$, then ***F'(X)=f(x).***

EXAMPLE

Let's prove the fundamental theorem of Calculus the way Cauchy did, using the mean value theorem and the additive property for definite integrals by using the limit definition of the derivative (also derived by Cauchy) on the function F(X).

Cauchy's Error

Cauchy falsely "proved" that an infinite series of everywhere continuous functions would be everywhere continuous.

Abel found the error, which was an example of a Fourier series. Abel's counter-example was:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots$$

This series represents the periodic function *x/2* on (-π,π) with period 2π.

Exercise

What's the error that Cauchy made? (HINT: what happens at $x = \pm\pi$?)

Check out this Desmos animation: https://www.desmos.com/calculator/skr4jpmknh

EXAMPLE

Let's find the Fourier Series for the periodic function *x/2* on [-π,π) with period 2π.

RECALL:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

Here,

$$A_0 = \frac{1}{2L} \cdot \int_{-L}^{L} f(x)\,dx$$

$$A_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n > 0$$

$$B_n = \frac{1}{L} \cdot \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n > 0$$