Class 19: **Tuesday November 7**

TITLE Cauchy, The Perfectionist
THIS READING: Katz, pp. 766-787; Boyer & Mertzbach, pp. 452-460

SUMMARY
We will review the life and work of Cauchy, widely regarded as one of the greatest mathematical geniuses of all time.

NEXT: Abel, Bolzano, Cauchy and Dedekind: The Rigor Debate in the Nineteenth Century
NEXT READING: Katz, pp. 766-787; Boyer & Mertzbach, pp. 526-538; Eves, 563-576

Augustin-Louis Cauchy (1789-1857)
Cauchy was one of the most prolific mathematicians of all-time, probably second behind Euler. He was a contemporary of Gauss (they died 2 years apart) but the breadth (and depth) of his contributions do not match that of the "prince of Mathematicians." Cauchy was born in 1789 and initially educated by his father, although he began attending *Ecole Polytechnique* in 1805, where he gained attention from Lagrange and Laplace after placing second out of nearly 300 in the entrance exam. In 1807, Cauchy went on to attend *École des Ponts et Chaussées* (Schools of Bridges and Highways) to become a civil engineer. However, his engineering career was short-lived and by 1816 Cauchy was a full professor of mathematics at *École Polytechnique*. While a professor, his lecture notes were published and became the foundation for almost all mathematician's understanding of central concepts in the Calculus and beyond.

Analytical Rigor

Cauchy published his most famous work, *Cours d'analyse de l'École Royale Polytechnique* (Analysis Course from the Royal Polytechnic School) in 1821. In it, he presents a rigorous approach to some of the central concepts in Calculus: limit, continuity, function, derivative and integral. He then used these definitions to improve some of the most important theorems in Calculus. Additionally, he brought order to the use of infinite series by devising tests that could determine their convergence (Cauchy Integral Test, Cauchy Root Test, etc). Cauchy also made important contributions to mechanics, elasticity and optics.

Cauchy and Complex Analysis

Cauchy contributed greatly to the analysis of a function of a complex variable and there are numerous results in this field which bear his name.

George Friedrich Bernhard Riemann (1826-1866) was a German mathematician who contributed greatly to analysis contemporaneously with Cauchy. He is most well-known for his contributions to differential geometry and calculus.

Cauchy-Riemann Equations

Given $Z(x+iy) = M(x,y) + i\,N(x,y)$ the Cauchy-Riemann equations (CRE) are

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x} \qquad\qquad \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

It turns out that the CRE are satisfied if the function $Z(x+iy)$ is analytic. Analyticity is a very important property of functions of a complex variable. Analyticity means that the function can be expanded into an infinite series that converges for all points z in an open disk $|z-z_0|<R$.

EXAMPLE

Let's use the definition of the derivative of a complex valued function of a complex variable, i.e. $Z(x+iy)=M(x,y)+iN(x,y)$ to derive the Cauchy-Riemann Equations.

Consider the expression

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{Z(z+\Delta z)-Z(z)}{\Delta z} &= \lim_{\Delta x+i\Delta y\to 0}\frac{Z(z+\Delta x+i\Delta y)-Z(z)}{\Delta x+i\Delta y}\\ &= \lim_{\Delta x+i\Delta y\to 0}\frac{M(x+\Delta x,y+\Delta y)+iN(x+\Delta x,y+\Delta y)-M(x,y)-iN(x,y)}{\Delta x+i\Delta y}\end{aligned}$$

and calculate the limits when $\Delta x=0$ and $\Delta y\to 0$ and when $\Delta y=0$ and $\Delta x\to 0$. We are assuming Z is analytic (so the derivative exists) which implies that the CREs will be satisfied.

Cauchy Integral Theorem

If the function *f(z)* is analytic, the value of the integral in the complex plane is independent of the path taken, and is zero on a closed path.

$$\oint_\gamma f(z)\, dz = 0.$$

Cauchy Integral Formula

Given an analytic function *f(z)*

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\, dz$$

Cauchy Differentiation Formula

The CIF leads immediately to the following result

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

Contour Integrals

A complex contour integral is barely distinguishable from a line or path integral in 2 dimensions.

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

Where where $z(t)=x(t)+ iy(t)$ for $a \le t \le b$ represents a path γ in the complex plane along which *f(z)* is being integrated.

When $z(a)=z(b)$ we write the integral as

$$\oint_\gamma f(z)\, dz$$

EXERCISE

Show that the integral of the complex function $f(z)=1/(z-z_0)$ along the circle $|z-z_0|=R$ traversed in a counter-clockwise direction is equal to $2\pi i$:

$$\oint_{|z-z_0|=R} \frac{1}{z-z_0} dz = 2\pi i$$

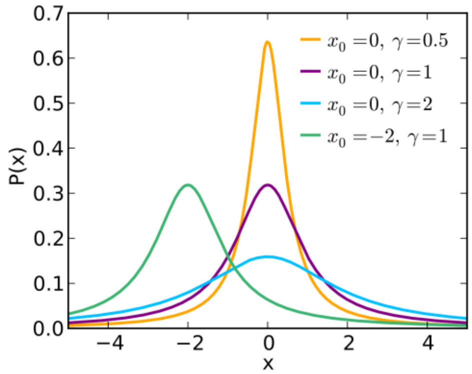
Cauchy's Other Contributions

The Cauchy Probability Distribution

The **Cauchy** distribution is a continuous probability distribution also known as the Lorentz distribution (by Physicists) given by:

$$f(x; x_o, \gamma) = \frac{1}{\pi\gamma\left[1 + \left(\frac{(x-x_0)}{\gamma}\right)^2\right]}$$

Where x_0 is the location of the peak and γ specifies the half-width of the curve at the maximum value $1/(\pi\gamma)$



Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality is an important and useful result in analysis. Given any two vectors **u** and **v** that are elements of an inner product space and an inner product <▪,▪> the Cauchy-Schwarz inequality is $|\langle \mathbf{u},\mathbf{v}\rangle|^2 \leq \langle \mathbf{u},\mathbf{u}\rangle\langle \mathbf{v},\mathbf{v}\rangle$.
When the inner product space is $\mathbb{R}^n$ and the inner product is the dot product $|\mathbf{u}\cdot\mathbf{v}|^2 \leq \mathbf{u}\cdot\mathbf{u}\ \mathbf{v}\cdot\mathbf{v}$.
One can also re-write Cauchy-Schwarz inequality in its most familiar form using vectors and their norms ||▪||:

$$|\vec{u} \cdot \vec{v}|^2 \leq ||\vec{u}||\ ||\vec{v}||$$

Cauchy-Euler Equations

Ordinary differential equations where the independent variable and the derivatives in a linear ODE are identical are known as Cauchy-Euler equations. For example, second-order Cauchy-Euler equations arise in solutions of Laplace's Equation and look like: $Ax^2y''+Bxy'+Cy=p(x)$. Selecting $y=x^r$ results in a quadratic equation in r which can be solved to find the particular solution of the homogeneous Cauchy-Euler equation $Ax^2y''+Bxy'+Cy=0$.

Cauchy Principal Value

The Cauchy Principal Value of an Improper Integral of the Third Kind is defined as

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$$

Which allows certain divergent integrals to be computed. Note: when $f(x)$ is an even function, i.e. $f(-x)=f(x)$ the principal value of the improper integral is the same as the value of the improper integral (if it converges).

Cauchy Residue Theorem

One of Cauchy's most enduring contributions to mathematics was his invention/discovery of the residue calculus, which allows one to evaluate numerous complex contour integrals much more simply, using something called a residue of a singularity. The residue, in some sense, quantifies the "strength" of a singularity in the Complex Plane and uses these values to compute the value of a closed contour integral that contains a finite number of singularities. The result is

$$\oint_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathbf{Res}(f; z_k)$$

Where the residue of a singularity (often called a "pole" of order *m*) at z_0 is given by

$$\mathbf{Res}(f; z_0) = \lim_{z \to z_0} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right\}$$

Applications of Complex Analysis to Evaluate Real Integrals

Recall the definition of $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Therefore, we can write $\cos(\theta)$ and $\sin(\theta)$ in terms of z.

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

We can use this information to convert integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$ into contour integrals on $|z| = 1$.

GROUPWORK

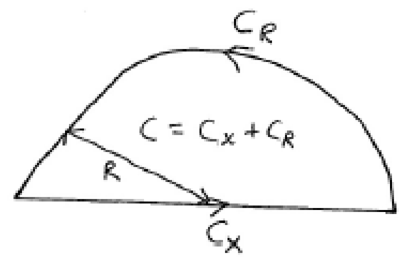
Rewrite the integral $\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta}$ in terms of z, using $z = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$.

Evaluate $\oint_{|z|=1} \frac{2}{z^2 + 4iz - 1} dz$

(HINT: the answer is $\frac{2\pi}{\sqrt{3}}$)

We can combine Cauchy Residue Theorem with Cauchy Principal Value to use Complex Integrals to evaluate some real integrals.

Let C be a contour consisting of C_R [i.e. a semi-circular arc of radius R centered at the origin from $(R,0)$ to $(-R,0)$] combined with C_x [the horizontal linear path from $(-R,0)$ to $(R,0)$ along the x-axis].



$$\int_{C_x} f(z)\,dz + \int_{C_R} f(z)\,dz = \oint_C f(z)\,dz$$
$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \oint_C f(z)\,dz$$

If we then take the limit as $R \to \infty$ and assume some boundedness conditions on $f(z)$ we can say that

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \oint_C f(z)\,dz = 2\pi i \sum Res(f)$$ (NOTE: only residues in the upper half-plane)

(since if the boundedness theorems above apply to $f(z)$ then $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$)

EXAMPLE

Show $\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$ Note that $\textbf{Res}(\frac{1}{1+z^4}, e^{\frac{i\pi}{4}}) = \frac{1}{4} e^{\frac{5\pi i}{4}}$ and $\textbf{Res}(\frac{1}{1+z^4}, e^{\frac{3i\pi}{4}}) = \frac{1}{4} e^{\frac{-\pi i}{4}}$.