Class 18: Thursday November 2

TITLE Gauss, Prince of Mathematicians
THIS READING: Katz 712; Boyer & Mertzbach 464-476; Eves, 476-479

SUMMARY
We will review the life and work of Gauss, widely regarded as one of the greatest mathematical geniuses of all time.

NEXT: Cauchy, the Perfectionist
NEXT READING: Katz, pp. 766-787; Boyer & Mertzbach, pp. 452-460

Carl Friedrich Gauss (1777-1855)
Gauss gets his nickname as the "Prince of Mathematicians" from the inscription "*Princeps mathematicorum*" that appeared on his tombstone. Gauss is one of the most precocious child prodigies of all time, with stories abounding of his mathematical prowess from as early as age 3. Gauss himself is said to have said "I could figure before I could talk" (Ball).

Proving the Fundamental Theorem of Algebra
Gauss' Ph.D. thesis at the age of 20(!) contained the first satisfactory proof of the fundamental theorem of algebra (i.e. "a polynomial equation with complex coefficients of degree $n>0$ must have at least one root in the complex plane"). Gauss returned to this theorem repeatedly in his life, publishing two more rigorous proofs in 1816 and another (the fourth!) in 1850 a few years before his death.

The main idea of Gauss' first proof is that a polynomial $f(z)=0$ where $z=a+ib$ is a complex variable will really turn into two polynomials $r(a,b)=0$ and $c(a,b)=0$ by isolating the real and complex parts of the equation.

In more modern times, rigorous proof of the fundamental theorem of algebra are believed to need to use topological results.

EXAMPLE
Show that a polynomial of the form $x^2+4i=0$ has a root of the form $a+bi$.

Number Theory
Disquisitiones arithmeticae (Investigations in Arithmetic) is Gauss' masterwork in number theory, published in 1801. Gauss said "Mathematics is the queen of the sciences, and the theory of numbers if the queen of mathematics" (Eves).

Disquisitiones was written in Latin, with a French translation appearing in 1807 but there was no English version unavailable until 1966(!)

Investigations in Arithmetic begins with the definition:

> **If a number *a* divides the difference between two numbers *b* and *c*, then *b* and *c* are said to be congruent, other incongruent; and *a* itself is called the modulus. Either of the numbers is called the residue of the other, in the former case, a nonresidue in the latter case.**

The terms and notation Gauss introduced in this book are still used today. i.e.

$$b \equiv c \pmod{a}$$

Gauss then went on to create a new form of algebra using congruency as a form of equality which has become very common in Number Theory to this day.

Exercise

Let us find solutions of the congruence $3x \equiv 51 \pmod{9}$. You should be able to find three less than 9.

Fundamental Theorem of Arithmetic
The idea of the fundamental theorem of arithmetic (also known as the unique factorization theorem) has been around since at least as far back as Euclid's *Elements*. The general idea is that every integer greater than 1 can be represented as a unique product of prime numbers (if one ignores order of the terms in the product).

Gauss includes a proof of the fundamental theorem of arithmetic in *Disquisitiones* as well.

Carl and Sophie
Sophie Germain (1776-1831) was able to master Calculus on her own and read Gauss's *Disquisitiones* when it came out and wrote the author letters under the pseudonym M. Le Blanc, to which he replied. However, due to her gender she was barred from attending the École Polytechnique in Paris. However, lecture notes were distributed freely to anyone who asked and Germain was able to study via letters to Lagrange and Legendre, who both taught there. Eventually, Lagrange became curious about M. LeBlanc and asked for a meeting, but was unperturbed to discover that his correspondent was female. Germain also inadvertently revealed her identity to Gauss, because fearing for an "Archimedean accident" when the French military was occupying the city that Gauss was living in, she wrote to the general in charge who personally saw to it that Gauss was safe, but was then confused as to how Mme. Germain could know of his existence. Gauss, too was pleased to discover M. Le Blanc was a female and according to Katz, wrote:

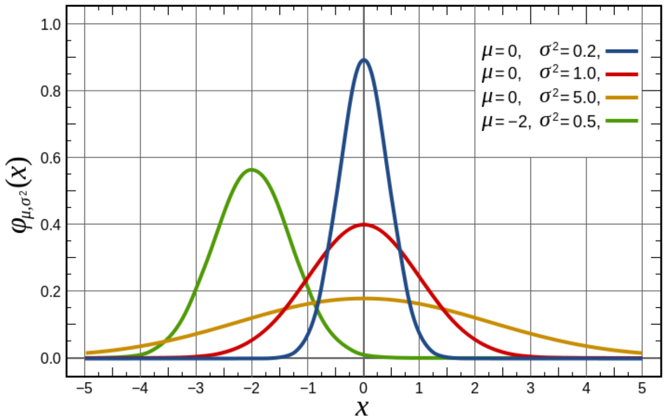
> **When a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men to familiarize herself with these thorny researches, succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of them, then without doubt she must have the noblest courage, quite extraordinary talents, and a superior genius.**

Gauss' Other Contributions (Partial List)
Gauss made contributions to many, many other scientific fields outside of mathematics. In Astronomy he produced an accurate prediction of the orbit of the "planet" Ceres using a technique which is still used today to track satellites called Gauss's Method. He also had contributions in physics, surveying, and geophysics.

Gauss's Law (in Physics)
This is a physical law which related the distribution of the electric charge to the electric field produced. Specifically, it says "the electric flux through any closed surface is proportional to the total electric charge enclosed by this surface."

Gauss's Theorem (The Divergence Theorem)
In multivariable calculus, the Divergence Theorem (also known as Gauss' Theorem) "relates the flux of a vector field through a closed surface to the divergence of the vector field in the volume enclosed by that surface."

The Gaussian Distribution
In probability and statistics, the Gaussian distribution is a type of continuous probability density function for a real-valued random variable which uses the Gaussian function with mean μ and standard deviation σ; such a function is said to be normally distributed, which is one of the most important and commonly used distributions in statistics. The graph of the Gaussian distribution over all values of the independent variable is known as ``the Bell curve" due to its distinctive shape.



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Figure 1: The Bell Curve (source: Wikipedia)

The Gaussian Function (or Gaussian)

The Gaussian or Gaussian function is a function of the form

$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}}$$

Where a is the height of the peak of the curve, b the location of the center of the peak and c the width of the the peak. Typically, when $b = \mu$ and $c^2 = \sigma^2$, and $a = 1/\sqrt{(2\pi\sigma^2)}$ the Gaussian function represents the normal distribution with normalized area under the curve $(-\infty,\infty)$ equal to 1.

Gaussian Quadrature

This is a clever method for approximating the area under a curve using a limited number of function evaluations.

Specifically, Gauss-Legendre Quadrature uses n function values and n weighted values to produce a formula that will evaluate a polynomial of degree $2n$-1 **exactly**.

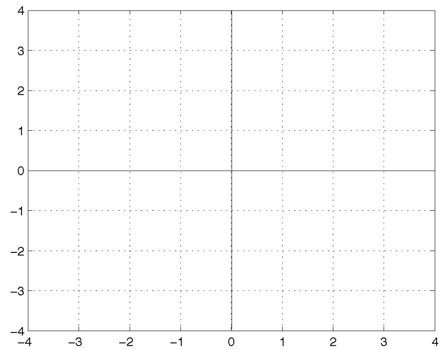
Gaussian Elimination (Row Reduction)

This is a method for solving a linear system of equations through the use of an ordered set of row operations which converts the augmented matrix representation of the linear system into reduced row echelon form which allows the reader to very easily find the solution.

Gaussian Integers
A Gaussian integer is a number of the form $a + b$i where a and b are integers. The set of Gaussian integers form something called the **imaginary quadratic field** as well as defining the **commutative ring**

$$\mathbb{Z}[\mathrm{i}] = \{a + b\mathrm{i} \mid a, b \in \mathbb{Z}\} \text{ where } \mathrm{i}^2 = -1$$

This means that the Gaussian integers can be visualized in the plane as points on the integer lattice.



Since $\mathbb{Z}$[i] is a commutative ring, the following axioms must be true for all elements in the set $\mathbb{Z}$[i] using the binary operations of addition + and multiplication ·.

1. $\mathbb{Z}$[i] is an **Abelian group** under addition +.

- **Addition is associative.** $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{Z}$[i]
- **Addition is commutative.** $a + b = b + a$ for all a, b in $\mathbb{Z}$[i]
- **Existence of an additive identity.** There is an element 0 in $\mathbb{Z}$[i] such that $a + 0 = a$ for all a in $\mathbb{Z}$[i]
- **Existence of an additive inverse**. For each a in R there exists $-a$ in $\mathbb{Z}$[i] such that $a + (-a) = 0$

2. $\mathbb{Z}$[i] is a **monoid** under multiplication ·.

- **Multiplication is associative.** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{Z}$[i]
- **Existence of a multiplicative identity.** There is an element **e** in $\mathbb{Z}$[i] such that $a \cdot \mathbf{e} = a$ and $\mathbf{e} \cdot a = a$ for all a in $\mathbb{Z}$[i]

3. Multiplication · is distributive under addition +.

- **Multiplication is left distributive.** $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all a, b, c in $\mathbb{Z}$[i]
- **Multiplication is right distributive.** $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in $\mathbb{Z}$[i]

DEFINITION
An **abelian group** is a set G with a group operation ▪ that satisfies closure, associativity, commutativity and the existence of an identity element and an inverse element.

DEFINITION
A **monoid** is a set G with a group operation ▪ that satisfies associativity and possesses an identity element.

DEFINITION
A **unit of a ring** is an element of the ring that has a multiplicative inverse. In other words, there exists another object in the ring which when the multiplication operation is applied to it and the unit the multiplicative identity is produced.

GROUPWORK

Confirm Show that $\mathbb{Z}[i]$ is an Abelian group.

EXERCISE

Show that $\mathbb{Z}[i]$ has four units: 1, -1, i and $-i$.

Gaussian Quadrature

In Gaussian quadrature one approximates the value of the area under a function *f(x)* on *x* in [-1,1] using two function value at x_1 and x_2 with weights c_1 and c_2 respectively. In other words,

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2)$$

There are four unknown constants x_1, x_2, c_1 and c_2 which provides **four** degrees of freedom and allows us to find these values so that evaluate the area of a polynomial with **four** unknown constants exactly.

Let's set up the Gaussian Quadrature equations and solve them exactly to obtain the general 2-point Gauss-Legendre quadrature formula.

$\boxed{\textbf{GroupWork}}$

Use Gaussian quadrature to evaluate $\int_{-1}^{1} 3x^3 + 2\,dx$ exactly!