Special Topics in Advanced Math: *History of Mathematics*

Math 395 Fall 2023 — Fowler 310 TR 1:30pm - 2:55pm

 — http://sites.oxy.edu/ron/math/395/23/

Class 16: **Tuesday October 26**

TITLE Euler, Part 2: The Basel Problem
THIS READING: Katz, pp 324-363; Boyer & Mertzbach, pp 223-281; Eves, pp 390-406

SUMMARY
We will continue our review of the life and work of Leonhard Euler, considered by many as the greatest mathematician of all time, and is certainly the most prolific.

NEXT: 18th Century Mathematicians: Agnesi, Bernouillis, Clairault, D'Alembert, Lagrange, Laplace & Legendre
NEXT READING: Katz, 712, 819-822, 834-838; Boyer & Mertzbach, 464-476

From Katz (p. 594):

BIOGRAPHY

Leonhard Euler (1707–1783)

Born in Basel, Switzerland, Euler showed his brilliance early, graduating with honors from the University of Basel when he was fifteen. Although his father preferred that he prepare for the ministry, Euler managed to convince Johann Bernoulli to tutor him privately in mathematics. The latter soon recognized his student's genius and persuaded Euler's father to allow him to concentrate on mathematics. In 1726 Euler was turned down for a position at the University, partly because of his youth. A few years earlier, however, Peter the Great of Russia, on the urging of Leibniz, had decided to create the St. Petersburg Academy of Sciences as part of his efforts to modernize the Russian state. Among the earliest members of the Academy, appointed in 1725, were Nicolaus II (1695–1726) and Daniel Bernoulli (1700–1782), two of Johann's sons with whom Euler had developed a friendship. Although there was no position in mathematics available in St. Petersburg in 1726, they nevertheless recommended him for the vacancy in medicine and physiology, a position Euler immediately accepted. (He had studied these fields during his time at Basel.)

In 1733, due to Nicolaus's death and Daniel's return to Switzerland, Euler was appointed the Academy's chief mathematician. Late in the same year he married Catherine Gsell with whom he subsequently had 13 children. The life of a foreign scientist was not always carefree in Russia at the time. Nevertheless, Euler was able generally to steer clear of controversies, until the problems surrounding the succession to the Russian throne in 1741 convinced him to accept the invitation of Frederick II of Prussia to join the Berlin Academy of Sciences, founded by Frederick I also on the advice of Leibniz. He soon became director of the Academy's mathematics section and, with the publication of his texts in analysis as well as numerous mathematical articles, became recognized as the premier mathematician of Europe. In 1755 the Paris Academy of Sciences named him a foreign member, partly in recognition of his winning their biennial prize competition 12 times.

Ultimately, however, Frederick tired of Euler's lack of philosophical sophistication. When the two could not agree on financial arrangements or on academic freedom, Euler returned to Russia in 1766 at the invitation of Empress Catherine the Great, whose succession to the throne marked Russia's return to the westernizing policies of Peter the Great. With the financial security of his family now assured, Euler continued his mathematical activities even though he became almost totally blind in 1771. His prodigious memory enabled him to perform detailed calculations in his head. Thus, he was able to dictate his articles and letters to his sons and others virtually until the day of his sudden death in 1783 while playing with one of his grandchildren (Fig. 17.7).

Boyer's *A History of Mathematics* says about Euler's 1748 work *Introductio in analysin infinitorum (Introduction to Analysis of the Infinite)*:

> **It may be fairly said that Euler did for the infinite analysis of Newton and Leibniz what Euclid had done for the geometry of Eudoxus and Theaetetus, or what Viete had done for the algebra of al-Khowarizmi and Cardan. Euler took the differential calculus and the method of fluxions and made them part of a more general branch of mathematics which has been known as "analysis"--the study of infinite processes. If the ancient *Elements* was the cornerstone of geometry and the medieval *Al jabr wa'l muqbalah* was the foundation stone of algebra, then Euler's *Introductio in analysin infinitorum* can be thought of as the keystone of analysis.**

Pierre-Simon Laplace (1749-1827) called Euler "the master of us all."

Barrow-Green's *History of Mathematics: A Source-Based Approach* says (202):

> It is of enormous importance for the history of mathematics that Euler published copiously — and at every level from the most advanced to the most elementary. He was not among the elitists, such as Descartes or Newton, who wrote for only a few and who are valued for that. He was among those who chose to speak to any audience, and did so very well. But more than that, he rode the 18th-century wave of publishing research and contributed greatly to its success.
>
> Euler had the gift of making his work look easy, even when the result is spectacular. His characteristic way of working was to start with a simple example, build up slowly and steadily from there, one step at a time, and keep going. This was surely an important part of what Euler wished to put across: doing mathematics using the systematic tools he presented may not be child's play, but brilliance is not necessarily required. It seems that he always did this — it was not something he slipped into when writing expository works, it was his habit in any substantial piece of research. His enormous productivity — Euler is by far the most prolific mathematician of all time — and the lucidity of his writing are two reasons why he had the influence that he did, as was his remarkable depth of insight.

Eves' *An Introduction to the History of Mathematics* says (435):

Euler's knowledge and interest were by no means confined to just mathematics and physics. He was an excellent scholar, with extensive knowledge of astronomy, medicine, botany, chemistry, theology, and oriental languages. He attentively read the eminent Roman writers, was well informed on both the civil and the literary history of all ages and nations, and showed a wide acquaintance with languages and with many branches of literature. Undoubtedly, he was greatly aided in these diverse fields by his uncommon memory.

Many glowing tributes have been paid to Euler, such as the following two made by the physicist and astronomer François Arago (1786–1853): ''Euler could have been called, almost without metaphor, and certainly without hyperbole, analysis incarnate.'' ''Euler calculated without any apparent effort, just as men breathe and as eagles sustain themselves in the air.''

The Basel Problem

Gottfried Wilhelm Leibniz (co-inventor of Calculus), the Bernouilli brothers and many other mathematicians were stumped by the problem of coming up with a closed form of the sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

Finding a closed form version of this sum (it was known to converge) and proving it was known as the Basel Problem.

Euler showed in 1735 that the solution was

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

Euler used two ideas:

1) The sum of the reciprocals of the roots of a polynomial equation written as $1 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n$ is equal to the negative coefficient of the linear term, i.e. $-c_1$. (This was a well-known result first proved by Rene Descartes) and

2) The "Maclaurin" series expansion of sin $z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ..$

Howard Eves explains:

> Then $\sin z = 0$ can (after dividing through by z) be considered as the infinite polynomial
>
> $$1 - z^2/3! + z^4/5! - z^6/7! + \cdots = 0,$$
>
> or, replacing z^2 by w, as the equation
>
> $$1 - w/3! + w^2/5! - w^3/7! + \cdots = 0.$$
>
> By the theory of equations, the sum of the reciprocals of the roots of this equation is the negative of the coefficient of the linear term—namely, $\frac{1}{6}$. Since the roots of the polynomial in z are $\pi, 2\pi, 3\pi, \ldots$, it follows that the roots of the polynomial in w are $\pi^2, (2\pi)^2, (3\pi)^2, \ldots$. Therefore
>
> $$\tfrac{1}{6} = 1/\pi^2 + 1/(2\pi)^2 + 1/(3\pi)^2 + \cdots,$$
>
> or
>
> $$\pi^2/6 = 1/1^2 + 1/2^2 + 1/3^2 + \cdots.$$

Let's work through PSP #1 *Euler's Calculation of the Sum of the Reciprocal Squares* together.

TASK 1

TASK 2

TASK 3

TASK 4

TASK 5

TASK 6

TASK 7

TASK 8

TASK 9

TASK 10

TASK 11