Special Topics in Advanced Math: *History of Mathematics*

Math 395 Fall 2023 | Fowler 310 TR 1:30pm - 2:55pm

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Class 12: **Thursday October 12**

TITLE Early 17th Century Stars & the March to Calculus: Mersenne, Fermat, Pascal, Galileo
THIS READING: Katz, 467-541; Boyer & Mertzbach, 300-348; Eves 318-325+346-366

SUMMARY
We will consider several prominent mathematicians of the Early 17th Century and how their contributions set the groundwork for the discovery/invention of Calculus.

NEXT: The Calculus of Newton and Leibniz
NEXT READING: Katz, pp 324-363; Boyer & Mertzbach, pp 223-281; Eves, pp 390-406

Marin Mersenne (1588-1648) was the link between some of the most famous scientific/mathematical names of all-time: Galileo, Pascal, Descartes, Huygens and Fermat. He corresponded with all of these mathematicians and has been described as a "walking scientific journal" (Katz 477) since generally people would send him a letter containing their latest results and he would forward a copy to each member of the group. This group was the core of what would become an institute of the French Academy in 1666.

René Descartes (1596-1650)
Descartes is known as "the father of modern philosophy" for the ideas published in his 1641 book *Meditationes*. His most famous book may be *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (Discourse on the Method of Reasoning Well and Seeking Truth in the Sciences) of 1637. He postulated that the entire universe is made up of matter in unending motion in vortices, and that all observed phenomena can be explained in terms of the forces exerted by adjacent matter. He is also well-known for the saying "I think therefore I am" (*Je pense donc je suis* or *cogito ergo sum*)

Descartes is also known as the father of analytic geometry (the intersection of algebra and geometry) for his use of what is now called the Cartesian coordinate system. In one of the appendixes of *Discourse* called *la géométrie* he demonstrated his new method. Descartes would take a geometric problem (like finding the tangent to a curve or the locus of points related to a number of lines). Descartes would convert the given geometric problem into an algebraic problem but then he would often use geometry to solve the algebraic problem. He saw geometry and algebra as two sides of the same coin.

Blaise Pascal (1623-1662)
Pascal was a child prodigy who presented his first paper at the age of 16(!) to Mersenne's group, where everyone was impressed with the youngster and his contribution except for Descartes, who was dismissive and insisted that it must have been written by Blaise's father, who was also mathematically inclined. He is most well-known for "Pascal's Triangle," although its existence had been known for at least six centuries at this point. Pascal did identify some new properties, and more importantly proved some of them using a technique which we now know as **mathematical induction**.

Pascal was very religious and gave up doing mathematics to concentrate on religious philosophy. During brief breaks from these pursuits he proved that a vacuum could exist and other scientific results which led his name being associated with the scientific unit of pressure, pascals.

Pierre de Fermat (1601-1665)
Fermat is Descartes main rival to mathematical ability during this era but he was not a professional mathematician or scientist but a lawyer and politician who merely dabbled in these subjects as an avocation or hobby. Fermat made great strides in analytic geometry as well, writing (a year before Descartes published *Discourse*):

> **Whenever in a final equation two unknown quantities are found, we have a locus, the extremity of one these describing a line, straight or curved.**

Basically, Fermat is describing the set of solutions of $f(x,y)=0$ as a representation of planar curves.

Fermat is most well-known for his contribution to number theory, primarily "Fermat's Last Theorem" and Fermat's "Little Theorem." In a letter (to Mersenne, of course) Fermat proved that 2^n-1 can not be prime if n is not prime. The primes which have the form 2^p-1 where p is prime are now known as **Mersenne primes**. Fermat's Little Theorem is "If p is any prime and a is any positive integer, then p divides a^p-1, or $a^p \equiv a$ (mod p).

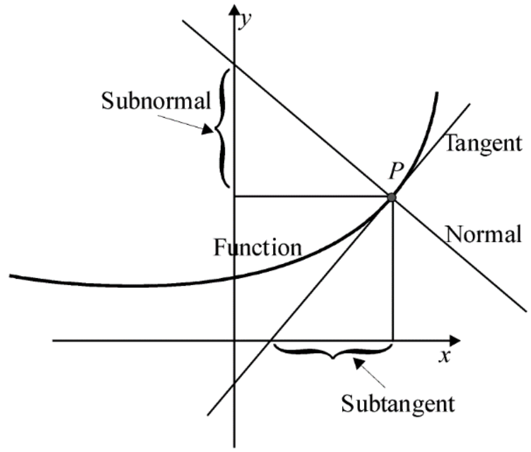
Fermat's Last Theorem is that there do not exist non-zero integers a, b, c and n (for n>2) such that $a^n+b^n=c^n$. Fermat was quite mischievous, and he claimed that he had a "truly marvelous demonstration [of this result] which this margin is too narrow to contain." The statement of the result is so simple that it captured the imagination of mathematicians everywhere but was not proven until 1995 by Andrew Wiles of Princeton using mathematician techniques that were not invented until hundreds of years after Fermat's death! Fermat was not flawless however. He repeatedly claimed that all number of the form $2^{2^\wedge n}$ were prime but Euler showed that $2^{2^\wedge 5}+1$ was not prime in 1732.

Galileo Galilei (1564-1642)
Galileo is most well-known for his *Dialogue Concerning the Two Chief World Systems* in 1637 which he presents the arguments for and against the Ptolemaic and Copernican views of the solar system through the dialogue between two students, one smart and one not so smart, and a teacher. By the end of the book, the mutually agreed upon view of the solar system is clearly resembles a heliocentric model of Copernicus. This led to conflict with the Catholic Church, which in 1633 banned the Dialogue and compelled Galileo to appear before an Inquisition in Rome and forced him to "confess his error." After the trial Galileo was sentenced to house arrest and prohibited from publishing any more work. Happily, he was able to publish *Discourses and Mathematical Demonstrations Concerning Two New Sciences* in 1638 and circumvent the publication ban by sending the manuscript to the Netherlands. Galileo made several discoveries related to kinematics which he was able to support with experiment and describe geometrically.

The Beginnings of Calculus

Prior to the work of Newton and Leibniz there was a great deal of work on early concepts that we now associate with calculus, some of which are:

- Infinitesimal analysis
- Tangents and Normals
- Maxima and Minima
- Areas and Volumes



Fermat's *Method of Adequality*

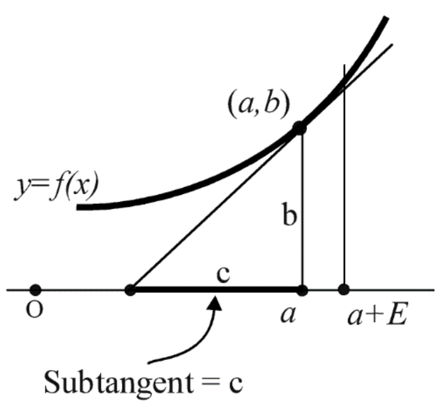
In 1629, Pierre de Fermat (1601-1665) wrote in *Method of Finding Maxima and Minima,* an algorithm he called the method of adequality which he used to find the extrema of functions and the slopes of tangents to curves.

Extrema

1) Given *f(x),* compute *f(x+E)* and adequate the two (set them equal to each other)
2) Cancel common terms and solve for *x* after removing any term that contains E.

EXAMPLE

Use Fermat's method of adequality to find the location of the maximum of
$f(x) = -x^2 + 3x - 2$



Fermat gives the formula ("the adequality") needed to find the tangent to a curve *y=f(x)* at *(a,b)* as

$$\frac{b}{c} = \frac{f(a+E)}{c+E}$$

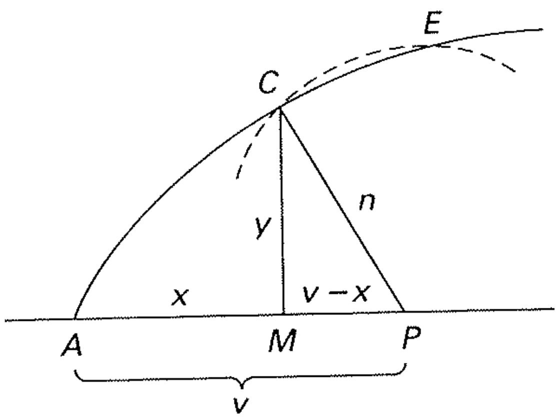
1) Cross multiply and cancel terms
2) Divide by E
3) Solve for *c* (ignoring terms with E or setting E=0)

Exercise

Use Fermat's method of adequality to define the tangent to $f(x) = x^2 - 2x + 3$ at $a=2$

René Descartes' Method of Normals

In *La géométrie* Descartes proposed a method to calculate normals to a curve at a specific point. Since tangents are perpendicular to normals, this was equivalent, and in Descartes mind, superior to the method of adequality of Fermat.



Descartes wants **P** to be the center of a tangent circle and thus sets up the equation

$$[f(x)]^2 + (v-x)^2 - n^2 = 0$$

Where

$$[f(x)]^2 + v^2 - 2vx + x^2 - n^2 = (x - x_0)^2 q(x) = 0$$

And solve v in terms of x_0 after balancing terms (often called the Method of Undetermined Coefficients)

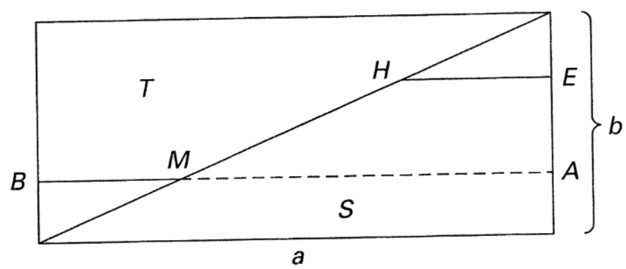
EXAMPLE

Use Descartes method of normals to obtain the slope of the normal to $y = x^2$ at the point (x_0, x_0^2) by obtaining the equations

$$\begin{aligned} a - 2x_0 &= 0 \\ b - 2x_0 a + x_0^2 &= 1 \\ ax_0^2 - 2bx_0 &= -2v \\ bx_0^2 &= v^2 - n^2 \end{aligned}$$

Integral Calculus

Bonaventura Cavalieri (1598-1647) was a student of Galileo's and the first to develop a coherent theory of working with "indivisibles" (infinitesimals). In 1635 he published *Geometry, Advanced in a New Way by the Indivisibles of the Continua* in which he introduced the concept of *omnes lineas* or "all the lines" of a plane figure F, which he denoted $\mathcal{O}_F(l)$. Cavalieri meant "the collection of intersections of the plane figure with a perpendicular plane moving parallel to itself from one side of the given figure to the other."



Each line segment *BM* in triangle T corresponds to a line segment *HE* in triangle S

$$\mathcal{O}_T(l) = \mathcal{O}_S(l)$$

Every segment BA from the rectangle is made up a line from triangle S and one from triangle T

$$\mathcal{O}_F(l) = \mathcal{O}_T(l) + \mathcal{O}_S(l)$$

Together these two equations imply that

$$\mathcal{O}_F(l) = 2\mathcal{O}_T(l)$$

Using modern calculus symbols, this result is equivalent to

$$ab = 2\int_0^b \frac{a}{b}t\ dt$$

Which becomes

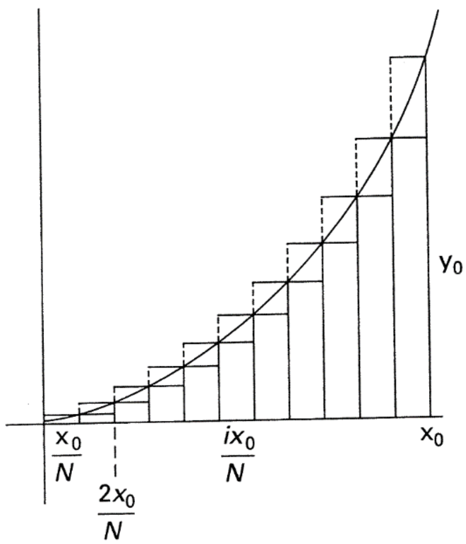
$$b^2 = 2\int_0^b t\ dt$$

By 1647 Cavalieri had been able to expand his method of indivisibles to compute the area under what was known as the "higher parabola" $y = x^k$:

$$\int_0^b x^k\ dx = \frac{1}{k+1}b^{k+1}$$

Katz claims that this result was known by many others working in the pre-Calculus mid 17th century such as Fermat, Pascal, Evangelista Torricelli (1608-1647) and Gilles Persone de Roberval (1602-1675)

The Area Under Any Higher Parabola



Divide the interval from x=0 to $x=x_0$ into N subintervals under the curve $y=px^k$ produces the expression for the sum of the areas of the rectangles

$$p\frac{x_0^k}{N^k}\frac{x_0}{N}+p\frac{(2x_0)^k}{N^k}\frac{x_0}{N}+\cdots+p\frac{(Nx_0)^k}{N^k}\frac{x_0}{N}=\frac{px_0^{k+1}}{N^{k+1}}\left(1^k+2^k+\cdots+N^k\right)$$

Which leads to inequalities for the exact area A

$$\frac{px_0^{k+1}}{N^{k+1}}(1^k+2^k+\cdots+(N-1)^k)<A<\frac{px_0^{k+1}}{N^{k+1}}(1^k+2^k+\cdots+N^k)$$

Considering that the difference between these two estimates goes to zero as N increases the exact area A is

$$\frac{px_0^{k+1}}{k+1}=\frac{x_0y_0}{k+1}$$

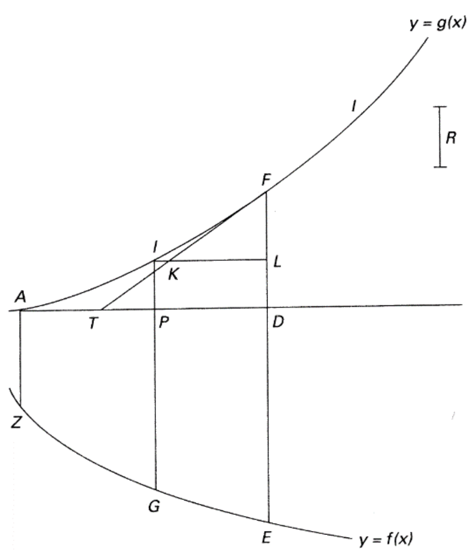
Both Roberval and Fermat used this method to compute this area.

Pascal used his knowledge of the Pascal's triangle to compute $\sum_{i=l}^{N} i^k$

and the result

$$\sum_{i=1}^{N-1} i^k<\frac{N^{k+1}}{k+1}<\sum_{i=1}^{N} i^k$$

Fermat was able to find the areas under "higher hyperbolas," i.e. curves of the form $y^m x^k = p$ in his *Treatise on Quadrature*. **John Wallis** (1616-1703), a British mathematician who taught at Oxford was able to compute the area under curves of the form $y = x^{p/q}$ where *p* and *q* are integers. **Isaac Barrow** (1630-1677) and **James Gregory** (1638-1675) understood versions of the Fundamental Theorem of Calculus. Barrow was Lucasian Professor of Mathematics at Cambridge while Newton attended there.



Barrow defines

$$Rg(x)=\int_a^x f(x)dx$$

$$g'(x)=\frac{g(x)}{t(x)}=\frac{f(x)}{R} \text{ or } \frac{d}{dx}\int_a^x f(x)\,dx=f(x)$$

$$\int_a^b Rf'(x)\,dx=Rf(b)-Rf(a)$$