Class 10: **Tuesday October 3**

TITLE Non-Western Perspectives, Part 1 (Mayan, Indian and Islamic Mathematics)
THIS READING: Katz, pp 230-321; Boyer & Mertzbach, pp 186-222

SUMMARY
We will begin to examine mathematic knowledge from non-European parts of the world.

NEXT: Non-Western Perspectives, Part 2: China, Africa and elsewhere
NEXT READING: Katz, pp 195-228, pp 364-380; Boyer & Mertzbach, pp 175-185; Gerdes (*Historia Mathematica*, Vol. 21, 345-376)

14th Century: Who Knew What?

GroupWork

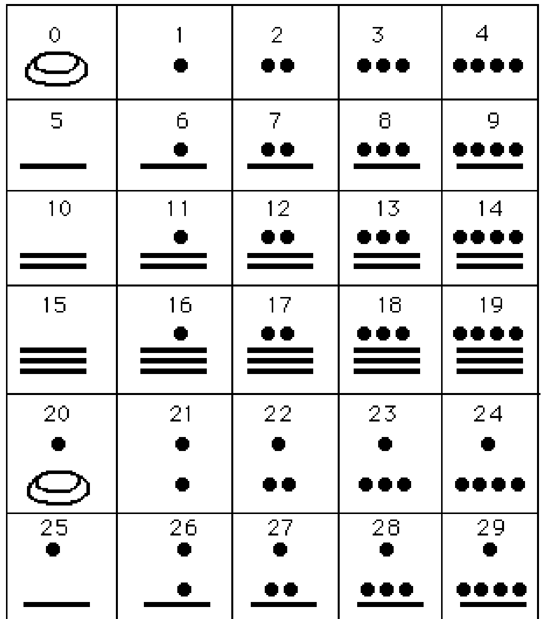
Review pages 365-368 of Katz and connect which sections of the world (China, India, "Islam," Europe) knew which areas of mathematics by the 14th century:

Trigonometry	China
Analytic Geometry	India
Algebra	Islam
Linear Congruences	Europe
Pascal's Triangle	Other
Calculus	

Why Did Modern Mathematics Develop in Europe?

What do you think about Victor Katz's argument for why modern mathematics developed in Europe?

The Mayans

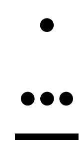


Mayan positional number system

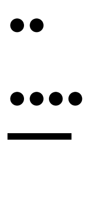
The Mayan civilization had a written language and a sophisticated civilization that flourished between the 3rd and 9th centuries in primarily what we call Central America nowadays. Their system of numeration was mainly a place value system with 20 as the base, but also used a grouping system with base 5. The ***third*** place in the number system would represent 360 (instead of 400) and then every place after that would represent 20 times the place before. The place value system was vertical with higher values at the top.

Example

Thus the Mayans would be represented the number 28 by



What would this Mayan number (shown below) represent?



The Incas

The Incas lived about 2000 miles south of the Mayans, in what is now known as Peru from around 1400 to 1560. Unfortunately, it is believed that they did not have a written language but they did have a logical numbering system using colored knots and cords on objects which are called ***quipus.*** These quipus used a base-10 place-value system. There's a picture of an Incan quipo on page 373 of Katz.

Islamic versus Arabic
The word Islamic is used to describe the contributions that occurred in the vast geographic region that was controlled by Muslims at one time.

Islamic Era
The Islamic era dates from around 622 CE when Mohammed left Mecca to go to Medina, often called "the Hejira." Mohammed died suddenly in Medina in 632 CE but this did not stop the rapid military expansion of Moslem forces to control an area as far west as Spain and as far east as India and parts of central Asia.

In 766, the caliph al-Mansur founded his capital at Baghdad, which became a commercial and intellectual center of the Arabian empire. The caliph al-Rashid established a library in Baghdad and began a program of collecting Greek manuscripts from the cities of Athens and Alexandria and translating them into Arabic. His successor, caliph al-Ma'mun, established the *Bayt al-hikma* (House of Wisdom) in Baghdad in an attempt to replicate the ancient research center at Alexandria.

The Father of Algebra
Mohammed ibn Musa al-Khwarizmi (c. 750-850 CE) is the most famous of the Arabic mathematicians and is sometimes called the "father of algebra."
(Note: **ibn** means "son of"; **abu** means "father of"; "**al**-X" means "from X" or "of X")

His work is so influential that he is credited with coining two words: algorithm and algebra. The word "algorithm" comes from a Latin description of al-Khwarizmi's work was described as "Dixit Algorismi" which became associated with doing arithmetic operations and turned into the English word algorithm.

The word "algebra" comes from "al-jabr" which appeared in the title of al-Khwarizmi most famous work *Al-kitab al-muhtasar fi hisab **al-jabr** wa-l-muqabala* (The Condensed Book on the Calculation of *al-Jabr* and *al-Muqabala*). *Al-jabr* was generally understood to refer to the operation of transposing a term from one side of the equation to another and *al-muqabala* is generally understood to mean comparing terms.

The work of al-Khwarizmi
In *The CondensedBook on the Calculation of al-Jabr and al-Muqabala* al-Khwarizmi systematically showed how to solve the kinds of equations which involved the square, the root of the square and the absolute number. There are six such kinds of equations

(Squares are equal to roots) $ax^2 = bx$
(Squares are equal to numbers) $ax^2 = c$
(Roots are equal to numbers) $bx = c$
(Squares and roots are equal to numbers) $ax^2 + bx = c$
(Squares and numbers are equal to roots) $ax^2 + c = bx$
(Squares are equal to roots and numbers) $ax^2 = bx + c$

Note that all the coefficients are positive and that zero was not a solution allowed by al-Khwarizmi, since his technique was basically geometric (like the Babylonians and Greeks).

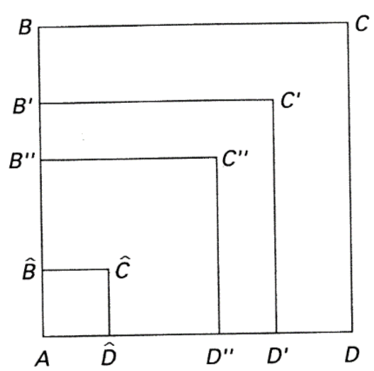
GroupWork

Write down the algebraic solution to all 6 types of al-Khwarzimi's equations. How many different problems would we classify these into today?

The Birth of Proof by Induction

Abu Bakr al-Karaji (d. 1019) gave the following result in the first decade of the 11th century in his book entitled *al-Fakhri* (*The Marvelous*)

$$\mathbf{1^3 + 2^3 + 3^3 + 4^3 + \ldots + 10^3 = (1+2+3+4+..+10)^2}$$



The square ABCD has side 1+2+3+…+10

The gnomon BCDD'C'B' has area 10^3

Area ABCD = Area AB'C'D' + Area Gnomon BCDD'C'B'

$(1+2+\ldots.10)^2 = (1+2+3+\ldots+10)^2 + 10^3$

Repeat the process with the next smaller square and gnomon

Finally the smallest square ABCD = smallest gnomon ÂBCD since $1=1^3$

How is this process (similar) different from what we know as mathematical induction today?

General Summation Relations
Egyptian mathematician Abu Ali al-Hasan ibn al-Hasan ibn al-Haytham (965-1039) derived the equation

$$(n+1)\sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k+1} + \sum_{p=1}^{n}\left(\sum_{i=1}^{p} i^k\right)$$

Ibn al-Haytham (also known as Alhazen) did not give the general form but for particular integers n=4 and k=1,2 3.

GroupWork

Let's show how we can use the Alhazen formulas to generate the following reasonably well-known formulas:

$$\sum_{i=1}^{n} i = \frac{n}{2}(n+1)$$

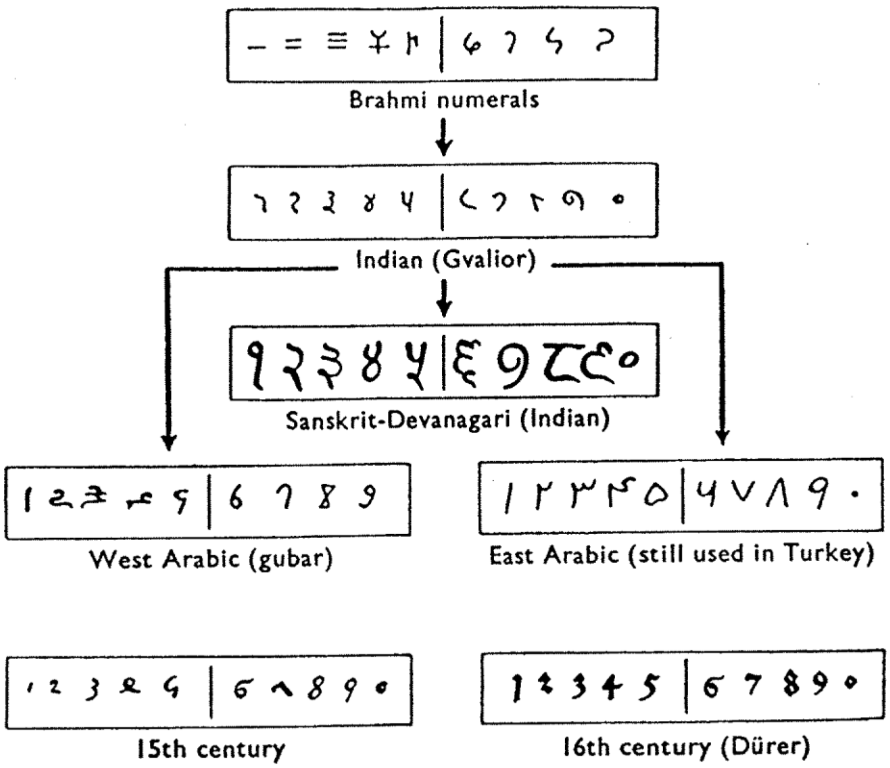
$$\sum_{i=1}^{n} i^2 = \left(\frac{n}{3}+\frac{1}{3}\right)n\left(n+\frac{1}{2}\right) = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n}{4}+\frac{1}{4}\right)n(n+1)n = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$$

$$\sum_{i=1}^{n} i^4 = \left(\frac{n}{5}+\frac{1}{5}\right)n\left(n+\frac{1}{2}\right)\left[(n+1)n-\frac{1}{3}\right]$$

Modern Numerals and Decimal Place System

The Indians are most well-known for first using only 10 symbols combined with a place-value system to represent numbers of all magnitudes. They also popularized the use of a symbol to represent zero.



Katz also mentions that Indians used words to represent individual numerals as well, such as sky for 0, moon for 1, eye for 2, fire for 3. They used a place system with units starting at the left.

EXAMPLE

moon-eye-sky-fire would be 3021. What would 2003 be?

The significance of the Indian contribution to the way we represent numbers today has often not been recognized but should not be forgotten. An eminent French mathematician, Pierre-Simon Laplace, said:

> It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity (Eves, 1988).

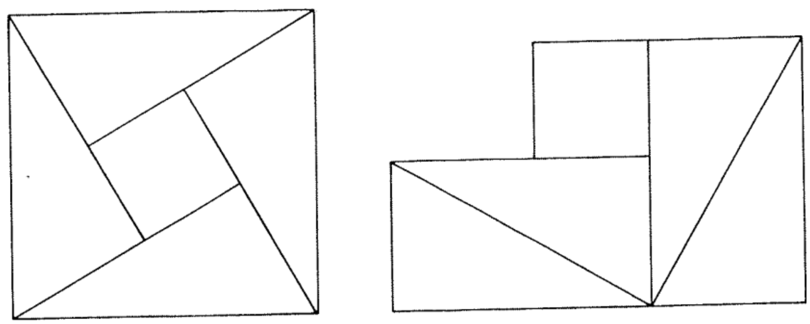
It should be noted that the Indians did not use decimal fractions; Islamic mathematicians developed their usage.

Indian Mathematics

Brahmagupta and **Bhaskara** are two of the most famous Indian mathematicians. They both flourished in the 7th century CE. There was a second mathematician with the same name Bhaskara later, so they are often denoted Bhaskara I and Bhaskara II.

Bhaskara I's Proofs of Pythagoras' Theorem

Bhaskara gave a pictorial "proof" of the Pythagorean theorem (which had clearly already been known for hundreds of years in India at the time because it appeared in older Indian writings called the *Sulbasutras*). He gave the following pictures and simply wrote "Behold!" (Eves, 1990).



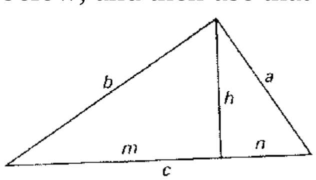
EXAMPLE

Let's show that given the sides of the triangle are (shorter side) *a* and (longer side) *b with hypotenuse* c, Bhaskara's proof is equivalent to showing that

$$c^2 = 4\left(\frac{ab}{2}\right) + (b-a)^2 = a^2 + b^2$$

Exercise

Bhaskara's 2nd Proof. First prove (all) the triangles are similar to obtain the expressions below, and then use that to show $c^2 = a^2 + b^2$



$$\frac{c}{b} = \frac{b}{m}, \quad \frac{c}{a} = \frac{a}{n}$$

The Etymology of "Sine"

Katz points out that the modern word "sine" is a result of incorrect translations of the word *jya-ardha* from Sanskrit, which means "chord-half." Aryhabhata abbreviated the term to *jya* or *jiva* which when translated into Arabic became *jiba* (which is not a word in Arabic). However, since Arabic is written without vowels, later Arabic readers saw the letters *jb* and assumed that it was representing the Arabic word *jaib* which means bosom or breast. Then, when the Arabic was translated into Latin in the 12th century the Latin word *sinus* was used (which means bosom). It was the Latin word *sinus* which became our modern English word **sine**!

Interestingly, Indian mathematicians knew power series approximations of several trigonometric functions, like

$$\cos s \approx 1 - \frac{s^2}{2} + \frac{s^4}{24} \qquad \sin s \approx s - \frac{s^3}{6} + \frac{s^5}{120}$$

Aryabhata (b. 476)
He is one of the earliest identifiable Indian mathematicians and wrote a book of mathematical results called *Aryabhatiya* where it is clear that he was able to apply the quadratic formula.

STANZA II, 19 *The desired number of terms minus one, halved, . . . multiplied by the common difference between the terms, plus the first term, is the middle term. This multiplied by the number of terms desired is the sum of the desired number of terms. Or the sum of the first and last terms is multiplied by half the number of terms.*[15]

This corresponds to the formula

$$S_n = n\left[\left(\frac{n-1}{2}\right)d + a\right] = \frac{n}{2}[a + (a + (n-1)d)].$$

where S_n is the sum of an arithmetic progression with constant difference d and first term a.

STANZA II, 20 *Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two. The result will be the number of terms.*

In other words, by considering the formula for S_n given in Stanza II, 19 as a quadratic in n we can show that it can be solved for n.

$$n = \frac{1}{2}\left[\frac{\sqrt{8S_n d + (2a-d)^2} - 2a}{d} + 1\right]$$

GroupWork
Consider the sequence of numbers a, $a+d$, $a+2d$, $a+3d$,…. What is the n^{th} term? Obtain a formula for the sum of the first n terms and confirm that your formula matches Aryabhata's given in Stanza II, 19.

Then use the quadratic formula on your answer to obtain an expression for the number of terms it takes to reach a fixed sum S_n and confirm that your formula matches Aryabhata's given in Stanza II, 20.