Class 4: Thursday September 7

TITLE Greek mathematical contributions: Logic, Paradox & Proof and Incommensurability
THIS READING: Katz, pp. 39-47; Boyer, pp. 67-70; Barrow-Green, pp. 95-109
NEXT READING: Katz, pp. 50-88; Boyer, pp. 90-108; Eves, pp. 140-155

SUMMARY
One of the singular achievements the Greeks are famous for giving the world is the notion of proof. We will explore the various contributions by Thales, Zeno, Aristotle and others in the area of mathematical knowledge before we consider the monumental achievements of Euclid.

Proof

Thales (c. 624-547 BCE) is widely described by historians as the "earliest mathematician" (Katz, 36) but what he was most well-known for is the idea that mathematical statements need to be **proved**. He was also famous for the idea that observable phenomena are governed by discoverable laws (which is basically the founding principle of all science).

Some propositions ascribed to Thales are:

> **Proposition.** A circle is bisected by any diameter.
> **Proposition.** The base angles of an isosceles triangle are equal.
> **Proposition.** The angles between two intersecting straight lines are equal.
> **Proposition.** Two triangles are congruent if they have two angles and the included side equal.
> **Proposition.** An angle in a semicircle is a right angle.

Pythagoras (c. 572-497 BCE) and his school/followers also used notions of proof to state and prove theorems about numbers and in plane geometry. Some examples of these are:

> **Proposition.** The sum of even numbers is even.
> **Proposition.** The sum of an even collection of odd numbers is even.
> **Proposition.** The sum of an odd collection of odd numbers is odd.
> **Proposition.** The side and diagonal of a square are incommensurable (i.e. cannot be used to measure each other using rational multiples).

Logic

Aristotle (384-322 BCE) is the person most closely associated with presenting a formalized logical system to mathematics. In fact, he insisted that logical argument is the *only* way to attain scientific knowledge.

The system is based around the notion of **axioms** (statements that are taken to be true without argument by everyone) and **postulates** (statements that are taken to be true in the current context of those axioms).

Aristotle promoted the use of **syllogisms** (consists of statements that when taken to be true lead to other statements to be true). For example, "If all monkeys are primates and all primates are mammals, then it follows that all monkeys are mammals."

NOTE

Aristotle's syllogisms were **NOT** the generally accepted form of argument used by the Greeks.

Examples of propositional logic

(1) *Modus ponens*

If p, then q.

p.

Therefore, q.

(2) *Modus tollens*

If p, then q.

Not q.

Therefore, not p.

(3) *Hypothetical syllogism*

If p, then q.

If q, then r.

Therefore, if p, then r.

(4) *Alternative syllogism*

p or q.

Not p.

Therefore, q.

Exercise

The oldest and most famous logical implications are known as *modus ponens* (Latin for "the way that affirms by affirming") and *modus tollens* (Latin for "the way that by affirming, denies"), often abbreviated as MP and MT, respectively.

Mathematically,
MP can be represented as $[(p \to q) \wedge p] \to q$
MT can be represented as $[(p \to q) \wedge \neg q] \to \neg p$

EXAMPLE

Let's verify these statements are valid arguments by using truth tables.

Modus Ponens $[(p \to q) \wedge p] \to q$

p	q	
T	T	
T	F	
F	T	
F	F	

The real power of these logical implications occurs when one thinks of the Boolean variables p and q as propositions, i.e. actual English statements that are either truth or false. For example,

p = "The sun is shining." q = "You will get a sunburn."

IF the sun is shining THEN you will get a sunburn.

The Sun is Shining.

CONCLUSION: You will get a sunburn.

Exercise

$$\textit{Modus Tollens}\ [(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$$

p	q	
T	T	
T	F	
F	T	
F	F	

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Paradox

Zeno of Elea (c. 495 - c. 430 BCE) presented four paradoxes which challenged the ways that centuries of mathematicians viewed the concept of infinity and approached the question of the continuity or discreteness of space-time itself.

Here are G. Donald Allen's descriptions of Zeno's paradoxes:

> **Dichotomy.** To get to a fixed point one must cover the halfway mark, and then the halfway mark of what remains, et cetera.
> **Achilles.** Essentially the same for a moving point.
> **Arrow.** An object in flight occupies a space equal to itself but that which occupies a space equal to itself is not in motion.
> **Stadium.** Suppose there is a smallest instant of time. Then time must be further divisible!

Here are Katz's descriptions of Zeno's Paradoxes (from original sources)

> **Dichotomy.** "asserts the non-existence of motion on the ground that that which is in locomotion must arrive at the half-way stage before it arrives at the goal."
> **Achilles.** "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."
> **Arrow. "**if everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless."
> **Stadium.** Consider two rows of bodies, each composed of an equal number of bodies of equal size. They pass each other as they travel with equal velocity in opposite directions. Thus, half a time is equal to the whole time.

Summary

> **Dichotomy.** Space is assumed divisible and time is assumed indivisible.
> **Achilles.** Space is assumed indivisible and time is assumed divisible.
> **Arrow.** Space and time are both assumed infinitely divisible.
> **Stadium.** Space and time are both assumed infinitely indivisible.

Aristotle in particular attempted to refute these paradoxes although most modern mathematicians believe that he was unsuccessful.