

## PSP #1: Basel Problem

TASK #1

$$a_{k+1} = r a_k$$

$$a_1 = r a_0$$

$$a_2 = r a_1 = r^2 a_0$$

Geometric Series if the proportion is constant  $r$ 

TASK #2 (a)

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} \quad \text{because } 3 < 4 \text{ so } \frac{1}{3} > \frac{1}{4}$$

$$\text{So } \frac{1}{3} + \frac{1}{4} > \frac{1}{2}$$

$$2(b) \quad \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \quad \text{since } 5 < 8, 6 < 8, 7 < 8$$

$$2(c) \quad \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$$

Since 9, 10, 11, 12, 13, 14, 15 are all less than 16, reciprocals are  $> \frac{1}{16}$ 

$$2(d) \quad \sum_{k=2^{n+1}}^{2^{n+1}+1} \frac{1}{k} > \frac{1}{2} \quad \text{Between } 2^{n+1} \text{ and } 2^{n+1}+1 = 2^n + 2^n = 2 \cdot 2^n$$

$$2^n + 1 \leq k \leq 2^{n+1} \quad \text{so } \frac{1}{k} > \frac{1}{2^{n+1}} \quad \text{when } k < 2^{n+1}$$

$$\sum_{k=2^{n+1}}^{2^{n+1}+1} \frac{1}{k} > \sum_{k=2^{n+1}}^{2^{n+1}+1} \frac{1}{2^{n+1}} = 2 \cdot \frac{1}{2^{n+1}} = \frac{1}{2}$$

$$2(e) \quad \text{Want to show that } \sum_{k=1}^n \frac{1}{k} > \frac{n}{2}$$

Use induction. When  $n=1$   $\sum_{k=1}^1 \frac{1}{k} = 1 + \frac{1}{2} = \frac{3}{2} > \frac{2}{2}$  ✓ BASE STEP

# PSP #1

$\frac{2}{4}$

INDUCTIVE: Assume  $\sum_{k=1}^{2^n} \frac{1}{k} > \frac{n}{2}$  and show  $\sum_{k=1}^{2^{n+1}} \frac{1}{k} > \frac{n+1}{2}$

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} = \underbrace{\sum_{k=1}^{2^n} \frac{1}{k}}_{\text{from } n^{\text{th}} \text{ case}} + \underbrace{\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}}_{\text{from (d)}} > \frac{n}{2} + \frac{1}{2} = \frac{(n+1)}{2}$$

STEP 2f

Basically part (e) is the modern representation of Oresme's statement.

If "each of which is longer than ~~the~~ half a foot" is the equivalent of the result from (d) and then when he says "there exist infinitely many parts... the total will be infinite" is equivalent to

$$\lim_{n \rightarrow \infty} A_{2^n} = \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \frac{1}{k} \Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2} = \infty$$

STEP 2g

Integral test has the same result (Harmonic Series diverges)

$$\int_1^{\infty} \frac{1}{k} dk = \ln(k) \Big|_1^{\infty} = \lim_{b \rightarrow \infty} \ln b = \infty$$

Integral Div  $\iff$  Series diverges

TASK 3

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Integral Test

$$\int_1^{\infty} \frac{1}{k^2} dk = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{k^2} dk = \lim_{b \rightarrow \infty} \left. -\frac{1}{k} \right|_1^b = \lim_{b \rightarrow \infty} 1 - \frac{1}{b} = 1$$

Integral converges  $\iff$  Series converges

# PSP #1

3/4

## TASK #4

The famous constant is  $\pi$  which is involved in the quadrature of the circle as  $\pi \cdot (\text{radius})^2$ .

## TASK #5

(a)  $f(s) = \sin s$ . The domain  $D$  of  $f$  is  $D = \{s \in \mathbb{R} \mid f(s) \in \mathbb{R}\}$ .

(b)  $f$  is discontinuous at  $s=0$  since  $f(s)$  is undefined. There are horizontal asymptotes as  $s \rightarrow \pm\infty$ ,  $f \rightarrow \pm\infty$ . No vertical.

(c)  $\sin s = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \frac{s^9}{9!} - \dots$

$$\sin s = 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!} + \frac{s^8}{8!} - \dots$$

$$\lim_{s \rightarrow 0} \sin s = \lim_{s \rightarrow 0} \left( 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \dots \right) = 1$$

## TASK #6

(a) Clearly Euler was talking about  $\pi = P$ .

(b) Perimeter of a circle of diameter  $d$  is  $\pi d = \pi$ .

(c) Zeros of this function equal the zeros of  $\sin s$  which are  $\pm\pi, \pm 2\pi, \pm 3\pi, \dots$  etc.

(d)  $0$  is not in the domain of  $\sin s$ .

## TASK #7

(a)  $x^2 - x - 6 = (x-3)(x+2) = (x-r_1)(x-r_2)$  so  $r_1 = 3, r_2 = -2$ .

$$1 + 1x - 1x^2 = \left(1 - \frac{x}{3}\right) \left(1 + \frac{x}{2}\right) = D$$

$$\left(1 - \frac{x}{3}\right) \left(1 - \frac{x}{2}\right) = 1 - \frac{x}{2} - \frac{x}{3} + \frac{x^2}{6} = \frac{r_1 r_2}{r_1 r_2} \left( \frac{r_1 r_2}{r_1 r_2} - (r_1 + r_2) \frac{x}{r_1 r_2} + \frac{x^2}{r_1 r_2} \right) =$$

$$\left(1 - \frac{x}{3}\right) \left(1 + \frac{x}{2}\right) = 1 - \frac{x}{2} + \frac{x}{3} - \frac{x^2}{6} = 1 + 1x - \frac{x^2}{6}$$

$$= \frac{1}{r_1 r_2} (x - r_1)(x - r_2)$$

TASK #8

(a) Let's use the Funky Factorization Theorem since function starts with constant 1

$$1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots$$

$$(b) = \left( \frac{1-x}{\pi} \right) \left( \frac{1+x}{\pi} \right) \left( \frac{1-x}{2\pi} \right) \left( \frac{1+x}{2\pi} \right) \left( \frac{1-x}{3\pi} \right) \left( \frac{1+x}{3\pi} \right) \dots$$

TASK #9

(a) Use the fact  $(1-a)(1+a) = (1-a^2)$  to write 8(b) as

$$\left( \frac{1-x^2}{\pi^2} \right) \left( \frac{1-x^2}{(2\pi)^2} \right) \left( \frac{1-x^2}{(3\pi)^2} \right) \dots$$

(b) Clearly the coefficient of  $x^2$  will have the form

$$-x^2 \left( \frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \dots \right)$$

TASK #10

$$(a) 1 - \frac{s^2}{3!} + \frac{s^4}{5!} - \frac{s^6}{7!} + \dots = \left( \frac{1-s^2}{\pi^2} \right) \left( \frac{1-s^2}{4\pi^2} \right) \left( \frac{1-s^2}{9\pi^2} \right) \dots$$

There are no "s" terms (linear in s) on either side

In  $(s^2)$

$$(b) \text{ LHS} = -\frac{1}{3!} \quad \text{RHS} = -\left( \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \dots \right)$$

$$(c) \frac{1}{3!} = \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \frac{1}{\pi^2}$$

$$\frac{\pi^2}{3!} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

TASK #11

Euler says it is doubtful that we can discover anything new about infinite series of reciprocal powers of integers but we don't even know whether  $\sum_{k=1}^{\infty} \frac{1}{k^3}$  is transcendental!