Special Topics in Advanced Math: *History of Mathematics*

Math 395 Fall 2023 Fowler 310 TR 1:30pm - 2:55pm

 `http://sites.oxy.edu/ron/math/395/23/`

Homework #11

[10 points total]

ASSIGNED: Tue Nov 14 2023 **DUE: Tue Nov 28 2023**

Informal Homework Responses

The first set of questions counts as one informal homework response. These problems can be worked on in groups with each student submitting their own solutions. They can be (**neatly!**) handwritten and can provide the "usual" amount of work to demonstrate how the solutions were obtained. Clearly indicate your answers.

Riemann Zeta Function and Bernoulli Numbers

These problems in Homework #11 will relate the Riemann Zeta Function, which is related to the Riemann Hypothesis (one of the most important unsolved problems in pure mathematics), and Bernoulli numbers, which are named after Jacob Bernoulli (1655-1705).

The Riemann Zeta function $\zeta(s)$ is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \text{ where } s \in \mathbb{Z}.$$

The n^{th} Bernoulli number B_n can be written in terms of ζ as

$$B_n = -n\zeta(1-n) \text{ for } n \geq 2$$

OR

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \text{ for } n \text{ odd}$$

OR

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}| \text{ for } n > 0$$

All odd Bernoulli numbers after $B_1 = 0.5$ are identically zero, i.e. $B_{2k+1} = 0$ for $k = 1, 2, 3, \ldots$ One can compute the Bernoulli numbers directly using the formula

$$B_n = \lim_{x \to 0} \frac{d^n}{dx^n} \left[\frac{x}{e^x - 1} \right] \tag{1}$$

OR one can obtain Bernoulli numbers by looking closely at the coefficients of Taylor expansions of certain functions. For example,

$$\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} x^{2n-1}, \qquad |x| < \frac{\pi}{2}. \tag{2}$$

1. **The (in)famous Ramanujan sum**. *[5 points]* **This problem can also be done as a Formal Homework.** When Ramanujan first wrote G.H. Hardy one of the results that amazed and perturbed the British mathematician was the following curious equation

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12} \tag{3}$$

We can show where this first example of a "Ramanujan sum" comes from by using the Riemann Zeta function.

   (a) *2 points.* Show that LHS (left-hand side) of Equation (3) is clearly equal to $\zeta(-1)$ and the RHS (right-hand side) is equal to $-\frac{B_2}{2}$

   (b) *2 points.* Use either one of the formulas in Equation (1) or Equation (2) to compute B_2.

   (c) *1 point.* Discuss your interpretation of the result that you have just proved given in (3). Why (or why not) does this equation make sense?

2. **Back to Basel.** *[5 points]* **This problem can also be done as a Formal Homework.** We previously discussed Euler's solution of the Basel problem, i.e. $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Using the Riemann Zeta function we can show how Euler was able to also give exact values for $\sum_{k=1}^{\infty} \frac{1}{k^4}, \sum_{k=1}^{\infty} \frac{1}{k^6}, \ldots, \sum_{k=1}^{\infty} \frac{1}{k^{2n}}$ for any value of n.

   (a) *1 point.* Use your previously computed value of B_2 to compute $\zeta(2)$ and confirm the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

   (b) *2 points.* Compute $\zeta(4)$ to find an exact value of $\sum_{k=1}^{\infty} \frac{1}{k^4}$. What Bernoulli number will you need to compute in order to obtain the answer? [HINT: Use Equation (2) to calculate this B_n.]

   (c) *2 points.* Obtain a general formula for computing the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$ like Euler did which involves B_{2n}. Use it to find the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^{14}}$. You can look up the value of the Bernoulli number you need instead of computing it this time!

(Optional) Formal Homework Responses

Choose ONE of the following problems as a formal homework response. This means that the solution is written up in LaTeX to generate mathematical symbols, uses complete sentences and is in narrative form; the work is done individually.

1. Do Problem 1 and include a long paragraph (50-100 words) describing Ramanujan sums in general and how they are used in other contexts.

2. Do Problem 2 and include a long paragraph (50-100 words) discussing Bernoulli numbers and they are used in another context.