# Special Topics in Advanced Math: *History of Mathematics*

Math 395 Fall 2023 Fowler 310 TR 1:30pm - 2:55pm

 `http://sites.oxy.edu/ron/math/395/23/`

## Homework #10

[20 points total]

**ASSIGNED: Tue Nov 7 2023** **DUE: Tue Nov 14 2023**

### Informal Homework Responses

The first set of questions counts as one informal homework response. These problems can be worked on in groups with each student submitting their own solutions. They can be (**neatly!**) handwritten and can provide the "usual" amount of work to demonstrate how the solutions were obtained. Clearly indicate your answers.

1. **Gaussian Quadrature.** *[5 points]* Consider the function $f(x) = 1 - \frac{1}{x^2}$ on $1 \leq x \leq 3$. Our goal is to use 2-point Gaussian Quadrature to obtain approximate values of $\mathcal{I}$, the area between this curve, $x = 1$, $x = 3$ and the $x$-axis.
   (a) *1 point.* Evaluate $\mathcal{I} = \displaystyle\int_1^3 1 - \frac{1}{x^2}\, dx$ exactly.
   (b) *1 point.* Transform $\mathcal{I}$ into an integral on $-1 \leq x \leq 1$ so that you can (HINT: you should come up with a formula for transforming an integral on $[a, b]$ to one on $[-1, 1]$.)
   (c) *1 point.* Use the 2-point Gaussian Quadrature formula $f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$ to approximate $\mathcal{I}$. Do you expect the answer to be exact? Why or why not?
   (d) *2 points.* In general, the $N$-point Gaussian Quadrature rule will have an error formula that looks like

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=1}^{N} c_i f(x_i) + \frac{2^{2N+1}(N!)^4}{(2N+1)[(2N)!]^3} f^{2N}(\xi)$$

   where $\xi$ is an unknown value in $[-1, 1]$ Use this formula to obtain an error bound for the approximation to $\mathcal{I}$ you found in (c). How does this formula explain why $N$-point Gaussian Quadrature will evaluate integrals of certain $k-th$ degree polynomials **exactly**? (HINT: You should be able to express $k$ in terms of $N$.)

2. **Trigonometric Integrals.** *[4 points]* We want to show that $\mathcal{I} = \displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$.
   (a) *2 points.* Show that if $z = e^{i\theta} = \cos\theta + i\sin\theta$ and $\cos\theta = \dfrac{z + 1/z}{2}$, $|z| = 1$ and $dz = iz\, d\theta$ then $\mathcal{I} = \displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$ can be re-written as $\mathcal{I} = \dfrac{1}{i}\displaystyle\oint_{|z|=1} \frac{2dz}{z^2 + 4z + 1}$.
   (b) *2 points.* Use Cauchy's Residue Theorem after finding the *relevant* poles and residues of $f(z) = \dfrac{2dz}{z^2 + 4z + 1}$ to show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$.
   Explain your answer and show all your work.

3. **Improper Integrals** *[4 points]* We want to show that $\mathcal{J} = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$.

   (a) *2 points.* Show that the expression $f(z) = \dfrac{1}{(1+z^2)^2}$ has poles of order 2 at $z = \pm i$ with residue equal to $\mp \dfrac{i}{4}$.

   (b) *2 points.* Use Cauchy's Residue Theorem to show that $\mathcal{J} = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$. Explain your answer and show all your work.

4. **Laguerre Polynomials.** *[7 points]* **This problem can also be done as a Formal Homework.** The goal of this problem is to demonstrate some properties of the Laguerre polynomials $L_n(x)$.

   (a) *2 points.* Use the Rodrigues' formula for the Laguerre polynomials, $L_n(x) = \dfrac{e^x}{n!}\dfrac{d}{dx^n}(x^n e^{-x})$ to confirm that $L_0(x) = 1$, $L_1(x) = 1 - x$, and to find $L_2(x)$.

   (b) *1 point.* To confirm the expression for $L_2(x)$ found in part (a) is correct, use the recurrence relation $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$.

   (c) *1 point.* Verify that your expressions for $L_0(x)$, $L_1(x)$, and $L_2(x)$ satisfy the Laguerre differential equation $xy'' + (1-x)y' + ny = 0$.

   (d) *2 points.* It turns out that there is an explicit formula for $L_n(x) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k$. Confirm that $L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$ using this explicit formula and one of the recursive formulas given above in parts (a) or (b).

   (e) *1 point.* Produce a graph of the first four Laguerre polynomials. What do you notice about $L_n(0)$?

## (Optional) Formal Homework Responses

Choose ONE of the following problems as a formal homework response. This means that the solution is written up in LaTeX to generate mathematical symbols, uses complete sentences and is in narrative form; the work is done individually.

1. Problem 4 on **Laguerre Polynomials**.

2. Write a 2-3 page (400-600 words) essay that compares and contrast the various approaches Euler and Cauchy had towards mathematical rigor (as we understand it in a modern content) and make an argument about why and how this consideration should be evaluated in deciding which of these mathematical giants should be considered the GOAT ("greatest of all time") of mathematicians