HW9

1

1 $x^2y'' + 6xy' + 6y = 0 \quad y(1) = 1 \quad y(-1) = 3$

Let $y = x^r \quad y' = rx^{r-1} \quad y'' = r(r-1)x^{r-2}$

$x^2 r(r-1)x^{r-2} + 6x\,rx^{r-1} + 6x^r = x^r(r^2 - r + 6r + 6) = 0$

$(r^2 + 5r + 6)x^r = 0 \Rightarrow (r+3)(r+2)x^r = 0 \Rightarrow r = -3, -2$

$y = Ax^{-3} + Bx^{-2}$

$x = 1, \quad 1 = A + B \qquad x = -1, \; y = -A + B = 3$

$\Rightarrow 2B = 4 \Rightarrow B = 2, \quad 2A = -2 \Rightarrow A = -1$

$y = -1x^{-3} + 2x^{-2}$ or $y = \frac{2}{x^2} - \frac{1}{x^3}$

$\begin{matrix} A + B = 1 \\ -A + B = 3 \end{matrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow$ Use Cramer's Rule

$$\Rightarrow A = \frac{\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{1\cdot 1 - 3\cdot 1}{1\cdot 1 - (-1\cdot 1)} = \frac{1-3}{1+1} = \frac{-2}{2} = -1 \checkmark$$

$$B = \frac{\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{1\cdot 3 - (-1)(1)}{2} = \frac{3+1}{2} = \frac{4}{2} = 2 \checkmark$$

HW #9

$\boxed{B}^2$

2. Consider $1+c_1x+c_2x^2=0$ as $(x-r_1)(x-r_2)=0$

$$x^2-(r_1+r_2)x+r_1r_2=0 = 1+c_1x+c_2x^2$$

$$\frac{x^2}{r_1r_2}-\left(\frac{r_1+r_2}{r_1r_2}\right)x+1=0$$

$$\frac{x^2}{r_1r_2}-\left(\frac{1}{r_2}+\frac{1}{r_1}\right)x+1 = c_2x^2+c_1x+1$$

So $c_1=-\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$ or $-c_1$ = sum of reciprocal of roots

ANOTHER WAY

$$x=\frac{-c_1\pm\sqrt{c_1^2-4c_2}}{2c_2} = -\frac{c_1}{2c_2}+\frac{\sqrt{c_1^2-4c_2}}{2c_2},\ -\frac{c_1}{2c_2}-\frac{\sqrt{c_1^2-4c_2}}{2c_2}$$

Sum of reciprocal of roots is

$$\frac{1}{-\frac{c_1}{2c_2}+\frac{\sqrt{c_1^2-4c_2}}{2c_2}}+\frac{1}{-\frac{c_1}{2c_2}-\frac{\sqrt{c_1^2-4c_2}}{2c_2}} = \frac{-\frac{c_1}{2c_2}-\frac{\sqrt{c_1^2-4c_2}}{2c_2}+-\frac{c_1}{2c_2}+\frac{\sqrt{c_1^2-4c_2}}{2c_2}}{\left(-\frac{c_1}{2c_2}+\frac{\sqrt{c_1^2-4c_2}}{2c_2}\right)\left(-\frac{c_1}{2c_2}+\frac{\sqrt{c_1^2-4c_2}}{2c_2}\right)}$$

$$=\frac{-\frac{c_1}{c_2}}{\left(\frac{c_1}{2c_2}\right)^2-\frac{c_1^2-4c_2}{4c_2^2}} = \frac{-c_1/c_2}{\frac{c_1^2}{4c_2^2}-\frac{c_1^2}{4c_2^2}+\frac{4c_2}{4c_2^2}} = \frac{-c_1/c_2}{1/c_2}=-c_1$$

#2

HW #9

3. We know

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \sum_{k=1}^{\infty} \frac{1}{4}\frac{1}{k^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{4}\cdot\frac{\pi}{6} = \frac{\pi}{24}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

So $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{3\pi^2}{24} = \frac{\pi^2}{8}$$

Or we could use Euler's method

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots$$

Let $z^2 = w$

$$\cos w = 1 - \frac{w}{2!} + \frac{w^2}{4!} - \frac{w^3}{6!} + \dots$$

$\cos z = 0$ when $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots, (2k-1)\frac{\pi}{2}$, $k \in \mathbb{Z}$

$z = (2k-1)\frac{\pi}{2}$, $k \in \mathbb{Z}$ are the roots $w^* = z^2 = (2k-1)^2\left(\frac{\pi}{2}\right)^2$

By Descartes sum of reciprocal roots rule

$$\sum_{k=1}^{\infty} \frac{1}{w^*} = \frac{1}{2!}$$

$$\sum_{k=1}^{\infty} \frac{1}{\left[(2k-1)\frac{\pi}{2}\right]^2} = \frac{1}{2!} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \left(\frac{\pi}{2}\right)^2\cdot\frac{1}{2} = \frac{\pi^2}{8}$$

本题 3/6

HW #9

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$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}\frac{1}{(2k-1)}^2$$

$$\text{So } \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{3\pi^2}{24} = \frac{\pi^2}{8}$$

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$$\cos w = 1 - \frac{w}{2!} + \frac{w^2}{4!} - \frac{w^3}{6!} + \dots$$

$$\cos \cancel{z} = 0 \text{ when } \cancel{z} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots, (2k-1)\frac{\pi}{2} \quad k \in \mathbb{Z}$$

$$\cancel{z} = (2k-1)\frac{\pi}{2}, \quad k \in \mathbb{Z} \text{ are the roots } w^* = z^2 = (2k-1)^2\left(\frac{\pi}{2}\right)^2$$

By Descartes sum of reciprocal roots rule

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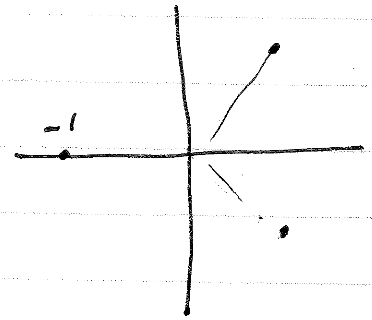
HW9

4. $(-1)^{1/3} = \left(e^{\pi i + 2n\pi i}\right)^{1/3} = e^{\pi i/3 + \frac{2}{3}n\pi i}$, $n = 0, 1, 2$

$= e^{\pi i/3}, e^{\pi i}, e^{\frac{5\pi i}{3}}$

$= e^{\pi i/3}, e^{\pi i}, e^{-\pi i/3}$

$= -1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$



$\log(-1) = \ln 1 + i\arg(-1)$

$= 0 + \pi i + 2n\pi i$

$= (2n+1)\pi i$

$\log\left(+\frac{1+i\sqrt{3}}{2}\right) = \ln\left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right) + i\arg\left(\frac{1+i\sqrt{3}}{2}\right)$

$= \ln 1 + i\frac{\pi}{3} + 2n\pi i = \left(2n + \frac{1}{3}\right)\pi i$

$\log\left(\frac{1-i\sqrt{3}}{2}\right) = \ln 1 + i\arg\left(\frac{1-i\sqrt{3}}{2}\right)$

$= 2n\pi i - i\frac{\pi}{3} = \left(2n - \frac{1}{3}\right)\pi i$

5. $V = \frac{1}{3}\pi r^2 h$, $S = \pi r\sqrt{r^2+h^2}$

$\frac{3V}{\pi r^2} = h \Rightarrow h^2 = \frac{9V^2}{\pi^2 r^4}$ So, $S = \pi r\sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}}$

$S = \sqrt{(\pi r)^2 r^2 + (\pi r)^2 \frac{9V^2}{\pi^2 r^4}} = \sqrt{\pi^2 r^4 + \frac{9V^2}{r^2}}$

$\frac{dS}{dr} = \frac{1}{2}\left(\quad\right)^{-1/2}\left(4r^3\pi^2 - 9V^2 \cdot \frac{2}{r^3}\right) = 0 \Rightarrow 4r^3\pi^2 = \frac{9V^2 \cdot 2}{r^3}$

$r^6 = \frac{9V^2}{\pi^2 2} \Rightarrow r^3 = \frac{3V}{\pi\sqrt{2}} = \frac{3}{\pi\sqrt{2}} \cdot \frac{\pi}{3} r^2 h = \frac{r^2 h}{\sqrt{2}}$

So $r = \frac{h}{\sqrt{2}}$ or $h = r\sqrt{2}$ $\therefore$ $S = \pi r\sqrt{r^2 + 2r^2} = \pi r^2\sqrt{3}$

HW9

5/6

5. $y = \frac{a^3}{x^2+a^2} = f(x)$ with $x = at$, $y = \frac{a}{1+t^2}$

(a) $\lim_{x\to\infty} \frac{a^3}{x^2+a^2} = 0$ $\lim_{x\to-\infty} \frac{a^3}{x^2+a^2} = 0$ so x-axis is an asymptote

$f(-x) = f(x)$ so y is an even function and symmetric about the y-axis

(b) $\int_{-\infty}^{\infty} \frac{a^3}{x^2+a^2}\,dx = a\int_{-\infty}^{\infty} \frac{1}{\left(\frac{x}{a}\right)^2+1}\,dx = a^2\int_{-\infty}^{\infty} \frac{1}{\left(\frac{x}{a}\right)^2+1}\,d\left(\frac{x}{a}\right)$

$= a^2 \arctan\left(\frac{x}{a}\right)\Big|_{-\infty}^{+\infty}$

$= a^2[\arctan(\infty) - \arctan(-\infty)]$

$= a^2\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \pi a^2$

The area of the circle with diameter a is $\pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$

(c) Volume of curve rotated about x-axis

$V = \pi\int_{-\infty}^{\infty} (f(x))^2\,dx = \pi\int_{-\infty}^{\infty} \left(\frac{a^3}{x^2+a^2}\right)^2 dx$

$= \pi a^6 \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx$

$\frac{1}{(x^2+a^2)^2} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2} \Rightarrow 1 = (Ax+B)(x^2+a^2) + Cx+D$

$1 = D + Ba^2$ $O(1)$

$0 = C + Aa^2$ $O(x)$

$0 = B$ $O(x^2)$

$0 = A$ $O(x^3)$

~~5~~ 6.6

WolframAlpha ☺

$$\int \frac{dx}{(a^2+x^2)^2} = \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + \frac{1}{2a^2}\,\frac{x}{x^2+a^2}$$

$$\text{so } \pi a^6 \int_{-\infty}^{\infty} \frac{1}{(a^2+x^2)^2}\,dx = \frac{\pi a^6}{2a^3}\arctan\left(\frac{x}{a}\right)\Big|_{-\infty}^{\infty} + \frac{\pi a^6}{2a^2}\left(\frac{x}{x^2+a^2}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{\pi a^3}{2}\left[\frac{\pi}{2} - -\frac{\pi}{2}\right] = \frac{\pi^2 a^3}{2}$$

(d) $$y = \frac{a^3}{a^2+x^2} \qquad \frac{dy}{dx} = \frac{-a^3}{(a^2+x^2)^2}\cdot 2x$$

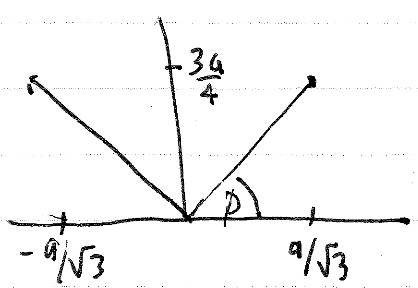
$$\frac{d^2y}{dx^2} = \frac{(a^2+x^2)^2\cdot -2a^3 - (-2a^3x)\,2x\,(a^2+x^2)\cdot 2}{(a^2+x^2)^4}$$

$$= \frac{(a^2+x^2)\,2a^3\left[-(a^2+x^2)+4x^2\right]}{(a^2+x^2)^4}$$

$$= -\frac{2a^3(3x^2+a^2)}{(a^2+x^2)^3} = 0 \quad \text{when } x = \pm a/\sqrt{3}$$

$$x^2 = a^2/3$$

$$y = a^3/(a^2+\tfrac{a^2}{3}) = 3a/4$$



$$\tan\theta = \frac{3a/4}{a/\sqrt{3}} = \frac{3\sqrt{3}}{4}$$