HW #8

1. $y = \frac{ax^{n-1}}{e+fx^n}$. $\int \frac{ax^{n-1}}{e+fx^n}\,dx = I$

Use u-substitution $u = x^n$ $du = nx^{n-1}dx$

$\frac{1}{n}du = x^{n-1}dx$

$I = \int \frac{a}{e+fu}\,\frac{du}{n}$

$= \frac{a}{n}\int \frac{du}{e+fu} = \frac{1}{n}\int \frac{a}{e+fu}\,du = \frac{1}{n}S = \frac{1}{n}\int z\,du$

where S is the area under the hyperbola $v = \frac{a}{e+fu}$

2. $y = \frac{1}{x}$ Use fluxions means replace y with $y+\dot{y}o$ and x with $x+\dot{x}o$

$y+\dot{y}o = \frac{1}{x+\dot{x}o} \Rightarrow \dot{y}o = \frac{1}{x+\dot{x}o} - y = \frac{1}{x+\dot{x}o} - \frac{1}{x}$

$\dot{y}o = \frac{x-(x+\dot{x}o)}{x(x+\dot{x}o)} = \frac{-\dot{x}o}{x^2+x\dot{x}o}$

$\dot{y} = \frac{-\dot{x}}{x^2+x\dot{x}o} \Rightarrow \frac{\dot{y}}{\dot{x}} = \frac{-1}{x^2+x\dot{x}o} = -\frac{1}{x^2}$ $(o\to 0)$

3. $y = x^x \Rightarrow \log y = x\log x$

$\cancel{d(\log y) = d(x\log x)}$

Leibniz' method is to replace y with $y+dy$ and x with $x+dx$

$\log(y+dy) = (x+dx)\log(x+dx)$

$\log\left[y\left(1+\frac{dy}{y}\right)\right] = (x+dx)\log\left[x\left(1+\frac{dx}{x}\right)\right]$

Recall: $\log(1+x) \approx x - \frac{x^2}{2} + \ldots$

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3 cont'd $\log y + \log\left(1+\frac{dy}{y}\right) = x\log\left(x\left(1+\frac{dx}{x}\right)\right) + dx\log\left(x\left(1+\frac{dx}{x}\right)\right)$

$$\log y + \left(\frac{dy}{y} + \ldots\right) = x\log x + x\log\left(1+\frac{dx}{x}\right)$$
$$+ dx\log x + dx\log\left(1+\frac{dx}{x}\right)$$
$$= x\log x + x\left(\frac{dx}{x} + \ldots\right) + dx\cdot\log x$$
$$+ dx\left(\frac{dx}{x} + \ldots\right)$$

But $\log y = x\log x$, so

$$\frac{dy}{y} = dx + dx\cdot\log x + \cancel{(dx)}$$
$$\frac{dy}{y} = dx(1+\log x)$$
$$dy = y\cdot dx(1+\log x) = x^x(\log(x)+1)dx$$

4. $\frac{dy}{dx} = \frac{1}{1+x}$, $y(0) = 0$

Let $y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

LHS $= \frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 = \sum_{k=1}^{\infty} k a_k x^{k-1}$

RHS $= \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 = \sum_{k=0}^{\infty} (-1)^k x^k$

$= \sum_{k=1} (-1)^{k-1} x^{k-1}$

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$$

$k a_k = (-1)^{k-1} \Rightarrow a_k = \frac{(-1)^{k-1}}{k}$

$k = 1, \ldots, \infty$

$$y = \sum_{K=1}^{\infty} \frac{(-1)^{K-1}}{K} x^K = \frac{1}{1}x + \frac{(-1)^1}{2}x^2 + \frac{(-1)^2}{3}x^3 + (-1)^3\frac{x^4}{4}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

Which is the same expansion as $\log(1+x)$.

5. $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{K=0}^{\infty} (-1)^K \frac{1}{2K+1}$$

So $\pi = 4 \sum_{K=0}^{\infty} \frac{(-1)^K}{2K+1}$

(b) Alternating Series Test for $\sum_{K=0}^{\infty} (-1)^K a_K$

(i) Show $a_{K+1} \leq a_K$ for all K

(ii) Show $\lim_{K\to\infty} a_K = 0$

(i) $\frac{1}{2(K+1)+1} = \frac{1}{2K+3}$ which is always less than $\frac{1}{2K+1}$ ✓

since $3 > 1 \Leftrightarrow 2K+3 > 2K+1$ and $\frac{1}{2K+3} < \frac{1}{2K+1}$

(ii) $\lim_{K\to\infty} \frac{1}{2K+1} = 0$ ✓

So $\sum_{K=0}^{\infty} \frac{(-1)^K}{2K+1}$ converges

(c) ~~Graph~~ Table N $4\left(\sum_{i=0}^{N} \frac{(-1)^i}{2i+1} - \pi\right)$

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5(e) $\varepsilon_1 = |s_n - \pi|$

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} |s_n - \pi| = \left|\lim_{n\to\infty} s_n - \pi\right| = \left|\lim_{n\to\infty} \varepsilon_n - \lim_{n\to\infty} \pi\right|$$

$$= |\pi - \pi| = 0$$

If you look at $\log \varepsilon_n$ versus $\log n$ you'll see a straight line with negative slope $= -1$

$$\varepsilon_n \sim \frac{1}{n} \Rightarrow \log \varepsilon_n \sim \log\left(\frac{1}{n}\right) \sim -\log(n)$$

There's a linear relationship between ε_n and n

So we say $\varepsilon_n = 0 + O\left(\frac{1}{n}\right)$ as $n \to \infty$