Special Topics in Advanced Math: *History of Mathematics*

Math 395 Fall 2023 — Fowler 310 TR 1:30pm - 2:55pm

 — http://sites.oxy.edu/ron/math/395/23/

Homework #7

ASSIGNED: Thu Oct 12 2023 — **DUE: Fri Oct 20 2023**

Informal Homework Responses

The first set of questions counts as one informal homework response. These problems can be worked on in groups with each student submitting their own solutions. They can be (**neatly!**) handwritten and can provide the "usual" amount of work to demonstrate how the solutions were obtained. Clearly indicate your answers.

1. (Adapted from **Katz, Chapter 15, p. 539, #8**.) Show that in modern notation, Fermats adequality method of finding the subtangent t to $y = f(x)$ by *adequating* the expressions $\dfrac{f(x+E)}{t+E}$ with $\dfrac{f(x)}{t}$ determines the subtangent t to be $t = f(x)/f'(x)$. Use Fermat's method to show that the subtangent t for the expression $f(x) = x^k$ is therefore $t = x/k$. Confirm this answer using modern Calculus techniques.

2. (**Katz, Chapter 15, p. 539, #17**.) Given that the volume of a cone is $\frac{1}{3}hA$, where h is the height and A the area of the base, use Kepler's method to divide a sphere of radius r into infinitely many infinitesimal cones of height r, and then add up their volumes to get a formula for the volume of the sphere.

3. (**Katz, Chapter 15, p. 540, #26).** Show that to find the length of an arc of the parabola $y = x^2$ one needs to determine the area under the hyperbola $y^2 - 4x^2 = 1$. [HINT: The length of an arc of the curve $y = f(x)$ on $[a, b]$ is given by $\int_a^b \sqrt{1 + (f'(x))^2}dx$]

4. (Adapted from **Eves, Chapter 10, p. 371, #10.2).** *This problem can also be done as a Formal Homework.* Consider the famous curve "the folium of Descartes" given by the equation $x^3 + y^3 = 3axy$ where a is a given parameter. (The line $x + y + a = 0$ is an asymptote.) The curve can be described parametrically in polar coordinates as $r = \dfrac{a \sec\theta \tan\theta}{1 + \tan^3\theta}$ for $0 \le \theta \le \pi$, or $x = \dfrac{3at}{1+t^3}, y = \dfrac{3at^2}{1+t^3}$ for $0 \le t < \infty$.

   (a) Show that the slope of the tangent line to the curve at any point is $\dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$

   (b) Show that the coordinates of the point at the edge of the loop are $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

   (c) Show that the equation of the tangent line at the edge of the loop is $x + y = 3a$.

   (d) Show that the area of the enclosed loop is $\dfrac{9a^2}{2}\displaystyle\int_0^{\pi/2} \dfrac{\sec^2\theta \tan^2\theta}{(1+\tan^3\theta)^2}d\theta$ which equals $\dfrac{3a^2}{2}$. [HINT: Use the substitution $u = \tan^3\theta$ to obtain and evaluate an improper integral!]

   (e) Use a computer (Desmos!) to produce a graph with the folium of Descartes, its tangent line and asymptote on the same axes for some positive value of a.

(Optional) Formal Homework Responses

Choose ONE of the following problems as a formal homework response. This means that the solution is written up in LaTeX to generate mathematical symbols, uses complete sentences and is in narrative form; the work is done individually.

1. (Adapted from **Eves, Chapter 10, p. 371, #10.2**) Consider the famous curve "the folium of Descartes" (in Figure 1) given by the equation $x^3 + y^3 = 3axy$ where a is a given parameter. (The line $x + y + a = 0$ is an asymptote.) The curve can be described parametrically in polar coordinates as $r = \dfrac{a \sec\theta \tan\theta}{1 + \tan^3\theta}$ for $0 \le \theta \le \pi$, or $x = \dfrac{3at}{1+t^3}$, $y = \dfrac{3at^2}{1+t^3}$ for $0 \le t < \infty$.

   (a) Show that the slope of the tangent line to the curve at any point is $\dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$

   (b) Show that the coordinates of the point at the edge of the loop are $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

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   (e) Use a computer (Desmos!) to produce a graph with the folium of Descartes, its tangent line and asymptote on the same axes for some positive value of a.

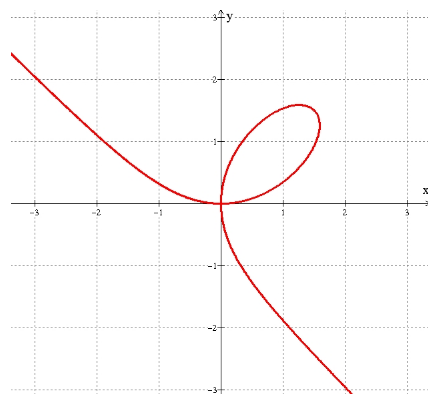

Figure 1: The folium of Descartes

2. (Adapted from **Katz, Chapter 15, p. 539, #7, #8 and #9.**) This problem is about expanding Fermat's "adequality" method to curves of the form $f(x, y) = c$. Note that if $(x + E, \tilde{y})$ is a point on the tangent line near (x, y), then $\tilde{y} = \frac{t+E}{t} y$. Fermat's method is (1) *adequate* $f(x, y)$ and $f(x + E, \frac{t+E}{t} y)$ (2) Solve this expression for t and then treat terms with E to be zero.

   (a) Show that using Fermat's modified adequality method given above for the Folium of Descartes $x^3 + y^3 - 3axy = 0$ the subtangent is $t = \dfrac{axy - y^3}{x^2 - ay}$.

   (b) Show that using modern notation, the modified version of Fermat's adequality method to $f(x, y) = c$ is equivalent to finding $t = -y(f_y/f_x)$ or $t = -y\left(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}\right)$

   (c) Recall Problem #2 from HW#4 of obtaining x^*, the x-intercept of the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point (x_0, y_0). Using the formula from part (b) one can obtain the expression for the subtangent $t = -\dfrac{ay_0^2}{b^2 x_0}$ and confirm that $x^* = a^2/x_0$.