Euler's Calculation of the
Sum of the Reciprocals of the Squares
PSP #1

TASK 1		**4**
TASK 2		**12**
TASK 3		**4**
TASK 4		**5**
TASK 5		**6**
TASK 6		**8**
TASK 7		**9**
TASK 8		**6**
TASK 9		**4**
TASK 10		**9**
TASK 11		**3**
Total		**70**

Comments:

Euler's Calculation
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A central theme of most second-semester calculus courses is that of *infinite series.* Simply put, to study infinite series is to attempt to answer the following question:

What happens if you add up infinitely many numbers?

How much sense humankind made of this question at different points throughout history depended enormously on what exactly those numbers being summed were. As far back as the third century BCE, Greek mathematician and engineer Archimedes (287 BCE–212 BCE) used his *method of exhaustion* to carry out computations equivalent to the evaluation of an infinite geometric series in his work *Quadrature of the Parabola* [Archimedes, 1897]. Far more difficult than geometric series are p-series: series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

for a real number p. Here we show the history of just two such series. In Section 1, we warm up with Nicole Oresme's treatment of the *harmonic series*, the $p = 1$ case.[1] This will lessen the likelihood that we pull a muscle during our more intense Section 3 excursion: Euler's incredibly clever method for evaluating the $p = 2$ case.

1 Oresme and the Harmonic Series

In roughly the year 1350 CE, a University of Paris scholar named Nicole Oresme[2] (1323 CE–1382 CE) proved that the harmonic series does not sum to any finite value [Oresme, 1961]. Today, we would say the series diverges and that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty.$$

His argument was brief and beautiful! Let us admire it below.

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[1]The *harmonic mean* of two numbers a and b is the quantity $2/(1/a + 1/b)$. Thus, in the harmonic series, each term is the harmonic mean of the term to its left and the term to its right, much like the terms of an arithmetic series or geometric series, *mutatis mutandis.*

[2]Nicole Oresme was an early Renaissance scholar who had a significant hand in reviving the works of antiquity. For instance, he translated Aristotle's *Ethics*, *Politics*, and *On the Heavens* from Latin into French. The work here stems from Oresme's study of Euclid's *Elements* [O'Connor and Robertson, 2003].

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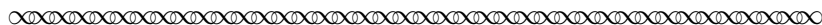
When quantities with unequal proportions (which are less than one) are added to some value, it is still possible that the sum will be infinite. But if it is that each term is proportional to the previous, the sum will be finite, ...

For example: if it we start with a measure of a foot, to which we first add a smaller portion (half a foot), then a third, and then a fourth, then a fifth and so on forever through all the numbers in order, I say that the sum will be infinite.

We shall see this as follows: if there exist infinitely many parts, each of which is longer than half a foot, then the total will be infinite. As is well known, $1/4$ and $1/3$ are more than a half; similarly, [the sum of] $1/5$ through $1/8$, and $1/9$ through $1/16$, etc. to infinity.[3]

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Task 1 Pay special attention to Oresme's claim that "if it is that each term is proportional to the previous, the sum will be finite," What kind of series was he talking about? Why must such a series have a finite sum?

Task 2 Since the question had been settled for series in which paris of consecutive terms had equal proportions, Oresme then looked at a series with unequal proportions: the sum of reciprocals of the positive natural numbers. Let us examine Oresme's argument in more modern form. In particular, define

$$a_n = \frac{1}{n}$$

to be the sequence of reciprocals and

$$A_n = \sum_{k=1}^{n} a_k$$

to be the corresponding sequence of partial sums.

(a) Explain why $1/3 + 1/4$ is greater than $1/2$ as claimed without actually calculating the sum of those two fractions. Rather, use the fact that $1/3 > 1/4$ and think about what would happen if you substituted $1/4$ for the $1/3$ in the sum $1/3 + 1/4$.

(b) In a similar manner, explain why $1/5 + 1/6 + 1/7 + 1/8$ is greater than $1/2$. Use the fact that each of the fractions $1/5$, $1/6$, and $1/7$ are greater than $1/8$.

(c) Extend this argument to explain why $1/9 + 1/10 + \cdots + 1/16$ is greater than $1/2$.

(d) Explain why

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} > \frac{1}{2}$$

for all natural numbers n.

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[3]The original Latin of the excerpt here is taken from pages 105 and 106 of *Quaestiones super Geometriam Euclidis* [Oresme, 1961]. Many thanks to Dominic Klyve (dominic.klyve@cwu.edu) for providing the translation.

(e) Use the above inequality to show that

$$A_{2^n} > \frac{n}{2}$$

for all natural numbers n.

(f) Consider Oresme's statement "if there exist infinitely many parts, each of which is longer than half a foot, then the total will be infinite." Why does the inequality $A_{2^n} > \frac{n}{2}$ justify his claim?

(g) A standard modern approach for analyzing the convergence/divergence of an infinite series is the *integral test*. Apply the integral test to the harmonic series. Does the conclusion agree with Oresme's?

2   The Sum of the Reciprocals of the Squares

A natural follow-up is to consider the sum of the reciprocals of squares! After all, a positive number less than 1 becomes smaller if you square the value; perhaps their sum could converge to a finite value even though the harmonic series diverges. Many mathematicians of the early 18th century attempted to compute this sum, written as

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

Task 3   Is the above summation even worth considering?

(a) Write the summation above in sigma notation.

(b) Use the integral test for convergence of an infinite series to conclude that it does in fact converge to a real number.

In the seventeenth century, a pair of mathematical brothers, Jacob (1655–1705) and Johann Bernoulli[4] (1667–1748), began to study this series. In 1689, in his *Tractatus de seriebus infinitis*, Jacob was able to prove the above series converged to a number less than 2 [Bernoulli, 1713]. However, the exact value of this series eluded the brothers. The solution was presented to the St. Petersburg Academy in 1735 by a brilliant former student of Johann named Leonhard Euler.[5]

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[4]The Bernoulli brothers were born into a successful Calvinist spice merchant family in Basel, Switzerland. The parents pushed Jacob into studying theology and philosophy. He defied them and switched to mathematics and astronomy later in life against their wishes. The parents then tried to push Johann into taking over the spice business; he despised this work and instead decided to follow his older brother into mathematics [O'Connor and Robertson, 1998a].

[5]Johann Bernoulli mentored a young Euler in his early days of mathematical studies. In his autobiography, Euler wrote, "In 1720, I was admitted to the university as a public student, where I soon found the opportunity to become acquainted with the famous professor Johann Bernoulli, who made it a special pleasure for himself to help me along in the mathematical sciences . . . which consisted in myself taking a look at some of the more difficult mathematical books and work through them with great diligence, and should I encounter some objections or difficulties, he offered me free access to him every Saturday afternoon, and he was gracious enough to comment on the collected difficulties, which was done with such a desired advantage that, when he resolved one of my objections, ten others at once disappeared, which certainly is the best method of making auspicious progress in the mathematical sciences." [Fellman, 2007, page 5]

3   Euler's Proof

At age 28, Leonhard Euler[6] (1707–1783) found the exact value of the sum! We now trace through his argument as it first appeared in print, in the 1740 paper *De Summis Serierum Reciprocarum* [Euler, 1740]. It was published in the *Commentarii academiae scientiarum imperialis Petropolitanae (Memoirs of the Imperial Academy of Sciences in St. Petersburg)*, the first journal of the St. Petersburg Academy of Science. It is worth noting, however, that the work was read to the academy five years earlier. Here we use Jordan Bell's translation [Euler, 2005].[7] As Oresme's work showed, the sums of reciprocals had been studied for hundreds of years at that point. Thus, Euler opened with the following remark.

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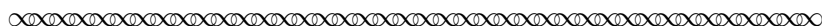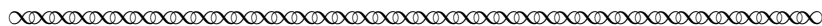
§.1. So much work has been done on the series of the reciprocals of powers of the natural numbers, that it seems hardly likely to be able to discover anything new about them.

<div align="center">. . .</div>

§.2. I have recently found, quite unexpectedly, an elegant expression for the sum of this series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.},$$

which depends on the quadrature of the circle. . . .

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Task 4 These days the word "quadrature" is often used interchangeably with the word "area" based on the ancient Greeks' perspective wherein finding the area of a region was accomplished by constructing a square (quadrilateral) that had the same area. According to Euler's statements above, on which famous constant will the summation depend?

Euler proceeded to state the relationship between the measure of an arc s on the unit circle and the sine of s, which he called y. Note that we typically refer to the input of sine as an angle rather than as an arc. On the unit circle, the radian measure of a central angle equals the length of the corresponding arc, since the unit circle would have a full circumference of 2π. Thus, radian measure of a central angle and the corresponding arc on the unit circle are in fact interchangeable.

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[6] Leonhard Euler was born in Basel, Switzerland to Marguerite (née Brucker) and Paul Euler, a Protestant minister who had attended Johann Bernoulli's lectures at University of Basel. Paul wished for his son to follow him into the ministry, but Johann persuaded Paul to allow Leonhard to study mathematics instead after witnessing his incredible potential for mathematics. The fact that the mathematicians who studied the evaluation of the sum of the reciprocals of the squares came from Basel has led to the problem being referred to as *The Basel Problem*. [O'Connor and Robertson, 1998b]

[7] The Mathematical Association of America hosts an incredible digital library known as *The Euler Archive*. This archive links to not only to the translation we use here, but to a massive quantity of Euler's writing, translated into many languages. Find it at `eulerarchive.maa.org`.

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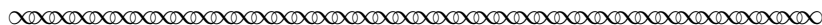
§.4. The first equation

$$y = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

thus expresses a relation between the arc and sine. For from it, from a given arc its sine can be determined, and likewise for a given sine the arc.
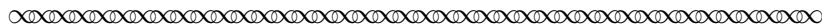
. . .

§.16. Now, if we let $y = 0$, the fundamental equation becomes

$$0 = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.},$$

and the roots of this equation give all the arcs whose sine is $= 0$. There is one minimal root $s = 0$, so that when the equation is divided by s it exhibits all the remaining arcs whose sine is $= 0$, so that these arcs are all the roots of the equation

$$0 = 1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

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Task 5 Euler discarded the root $s = 0$ in his work above. Let us examine his decision to do so in a bit more modern notation.

(a) Consider the function $f(s) = \frac{\sin(s)}{s}$. What is the domain of f?

(b) Where is f discontinuous? Does the graph of f have any vertical asymptotes? Why or why not?

(c) Recall from your first-semester calculus course your strategies for evaluating limits; often when division by zero was encountered, a good strategy for removing that discontinuity was to factor the numerator and reduce the fraction. Take the function $f(s) = \frac{\sin(s)}{s}$ and rewrite the numerator as a power series centered at zero. Replicate this "factor and cancel" trick to remove the discontinuity. That is, calculate
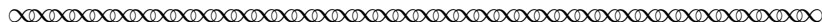
$$\lim_{s \to 0} \frac{\sin(s)}{s}$$

not by using L'Hôspital's Rule or a special limit identity, but rather by using the power series for sine as Euler did!

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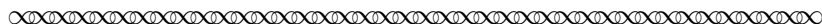
§.16. Of course these arcs whose sine is $= 0$ are

$$p, -p, 2p, -2p, 3p, -3p, \text{etc.},$$

of which the second of each pair is negative, which the equation itself also tells us, because the dimensions of s are even.

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Task 6   (a) Based on knowing the zeros of the sine function, what number would you guess Euler was talking about when he uses the symbol p above?

(b) In §.11. of the same document, Euler introduced the symbol p in the phrase "where p denotes the total perimeter of a circle whose diameter is $= 1$." What does this phrase imply the value of p is? Does it agree with your conclusion from part (a)?

(c) Why are these the zeros of the function $1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \cdots$, as he claimed?

(d) Why is the number 0 not on Euler's list of arcs? Isn't the sine of 0 equal to 0?

Euler concluded §.16. with a very familiar task, polynomial factorization, presented in a somewhat unfamiliar way.

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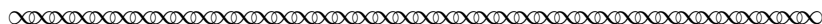
Hence the divisors of this equation will be

$$1 - \frac{s}{p}, 1 + \frac{s}{p}, 1 - \frac{s}{2p}, 1 + \frac{s}{2p}, \text{etc.}$$

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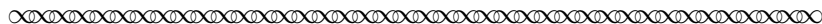
What is happening here may look odd at first, but it is really just a rewritten version of the Factor Theorem from any introductory algebra class. We usually state the Factor Theorem as follows:

Familiar Factor Theorem.
A polynomial $f(x)$ has a root $x = r$ if and only if it is divisible by $(x - r)$.

But, one could take the divisor $(x - r)$ and factor out the constant $-r$, producing instead the expression

$$(x - r) = -r\left(\frac{x}{-r} + 1\right) = -r\left(1 - \frac{x}{r}\right).$$

Factoring out the constant $-r$ does not affect the divisibility of those polynomials; the polynomial $f(x)$ is divisible by $(x - r)$ if and only if it is divisible by $\left(1 - \frac{x}{r}\right)$. Thus, an equivalent but funky restatement of the Familiar Factor Theorem would be this:

Funky Factor Theorem.
A polynomial $f(x)$ has a root $x = r$ if and only if it is divisible by $\left(1 - \frac{x}{r}\right)$.

Why have two versions of the same theorem? Well, the first version is very convenient if the leading coefficient of the polynomial is 1. In this case, if $f(x)$ has roots $r_1, r_2, \ldots, r_n$, then

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n),$$

since upon expansion, the right-hand side will have leading term x^n and thus a leading coefficient of 1. However, instead of having a leading coefficient of 1, suppose $f(x)$ has a constant term of 1. In this case, the Funky Factor Theorem is more convenient! In particular, it tells us

$$f(x) = \left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right) \cdots \left(1 - \frac{x}{r_n}\right).$$

This works because when the above product is multiplied out, the constant term is 1. For fear of a relatively simple idea being obfuscated by formalism, we work out a small concrete example.

Task 7   (a) Factor the polynomial $x^2 - x - 6$. What are the roots? Call these numbers r_1 and r_2.

(b) Show that the polynomial $1 + \frac{1}{6}x - \frac{1}{6}x^2$ (which is the same quadratic, just divided by -6) has the same roots as the previous polynomial by just plugging in r_1 and r_2 for x and verifying the output is zero in each case. Use these roots to factor $1 + \frac{1}{6}x - \frac{1}{6}x^2$ into the form $\left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)$.

(c) In each case, multiply out your factorization to see that it is correct.

Task 8   Let us now revisit Euler's statement in which he claimed to know "the divisors of this equation".
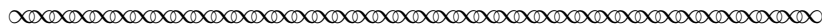
(a) We have the polynomial[8]

$$1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \cdots$$

that we wish to factor. Which version of the Factor Theorem should we use, the Familiar or the Funky? Why?
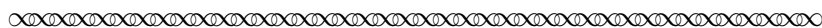
(b) Write out the factorization that this produces, using the roots from Task 6.

Euler proceeded with a simple but clever simplification.

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and by combining each of these pairs of divisors together it will be

$$1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

$$= \left(1 - \frac{s^2}{p^2}\right)\left(1 - \frac{s^2}{4p^2}\right)\left(1 - \frac{s^2}{9p^2}\right)\left(1 - \frac{s^2}{16p^2}\right) \text{etc.}$$

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[8] Technically this is a power series and not a polynomial, since the degree is infinite. However, Euler had great intuition regarding the generality of the structure of mathematics; he was confident we could factor an infinite degree polynomial in the same way that we could factor a finite degree polynomial, and proceeded accordingly!

Task 9 (a) Which pairs of divisors did he join together? What famous identity from algebra did Euler apply to produce that right-hand side?

(b) When you look at those denominators and think about the question we had set out to answer, do the hairs on the back of your neck tingle a little? If not, remind yourself what it is exactly that we are trying to evaluate, and then see if they do.

Euler only stated his next step in words; we will carry out the corresponding algebra.
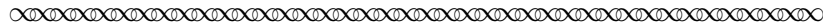
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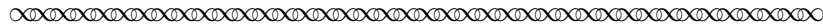
§.17. Now it is evident from the nature of the equations, that the coefficient of ss, or $\frac{1}{1\cdot2\cdot3}$, is equal to

$$\frac{1}{p^2} + \frac{1}{4p^2} + \frac{1}{9p^2} + \frac{1}{16p^2} + \text{etc.}$$

$$\cdots$$

§.18. Hence from these the following sums are thus derived:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = \frac{p^2}{6}.$$

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Task 10 Let us retrace the algebra that Euler glossed over above.

(a) Take Euler's infinite factorization from before,

$$1 - \frac{s^2}{3!} + \frac{s^4}{5!} - \frac{s^6}{7!} + \cdots = \left(1 - \frac{s^2}{p^2}\right)\left(1 - \frac{s^2}{4p^2}\right)\left(1 - \frac{s^2}{9p^2}\right)\left(1 - \frac{s^2}{16p^2}\right)\cdots,$$

and begin to multiply out the right-hand side. Specifically, calculate the constant term as well as the coefficients of s and s^2.

(b) Set the degree two coefficient of the right-hand side equal to the degree two coefficient of the left-hand side.[9]

(c) Multiply both sides by p^2 and declare victory!

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[9] Euler did not worry about convergence of this power series or of this infinite product. He never tried to plug in any values for s, so issues of convergence are, in a certain sense, irrelevant! Rather, he is interested in extracting information by studying the coefficients of the power series. This approach has been developed into a theory that we today call *generating functions*. In this discipline, one treats power series as formal algebraic objects (which in this context get called generating functions) in order to extract information about the coefficients. In Herbert Wilf's classic text *generatingfunctionology* [Wilf, 2005], he starts the first page of the first chapter by saying "A generating function is a clothesline on which we hang up a sequence of numbers for display." This is exactly the perspective Euler took here. In this work, he never used the power series for $\frac{\sin(s)}{s}$ to calculate values of that function for numerical values of s. Rather, he expressed the degree two coefficient of that power series in two different ways and equated the expressions.

4   Epilogue

In the same paper, Euler provided a method to extend this technique to any even power and executed the method up to exponent 12 in §.18. Just for fun, here are the values!

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{\pi^{10}}{93555}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{12}} = \frac{691\pi^{12}}{638512875}$$

With slightly different techniques, Euler proceeded to publish the values of the even-powered sums up to exponent 26 in his 1748 *Introductio in Analysin Infinitorum*, Volume 1 [Euler, 1748, page 131].

It is also worth noting that besides the harmonic series, no one has ever computed the sum for a single odd power! For instance, there has been little progress in understanding the sum of reciprocals of the cubes. The number

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is known as *Apéry's constant*. In 1979, Roger Apéry (1916–1994) published a proof that it is *irrational*, [Apéry, 1979] meaning that it cannot be expressed as a ratio of integers. However, it is still unknown whether or not that value is *transcendental*! That is to say, in theory, there could be some polynomial equation with integer coefficients which has Apéry's constant as a root.

Task 11 In light of what you just read about Apéry's constant, what is ironic about Euler's opening statement?

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Let's work through PSP #1 *Euler's Calculation of the Sum of the Reciprocal Squares* together.

TASK 1

TASK 2

TASK 3

TASK 4

TASK 5

TASK 6

TASK 7

TASK 8

TASK 9

TASK 10

TASK 11