Senior Colloquium: *History of Mathematics*

Math 400 Spring 2020

Fowler 310 T 1:30pm - 2:55pm
`http://sites.oxy.edu/ron/math/400/20/`

QUIZ : April 14, 2020

NAME:______________________ **TOTAL**______/50

This quiz is intended to take 60 minutes. There are 4 pages and 4 parts.

MATH PART I: (15 points)

The goal of this problem is to show that $\mathcal{I} = \int_0^{2\pi} \frac{d\theta}{b+\cos\theta} = \frac{2\pi}{\sqrt{b^2-1}}, b > 1.$

(a) Use the substitution $z = e^{i\theta}$ to show that $\mathcal{I}$ can be re-written as $\mathcal{J} = \frac{1}{i}\oint_{|z|=1} \frac{2\,dz}{z^2+2bz+1}$

$dz = ie^{i\theta}d\theta = iz\,d\theta$

$\frac{dz}{iz} = d\theta$

$\cos\theta = \left(z + \frac{1}{z}\right)\frac{1}{2}$

$J = I$ because one is basically doing integration by substitution to go from $I \to J$

$$I = \oint_{|z|=1} \frac{dz}{iz} \cdot \frac{1}{b+\left(z+\frac{1}{z}\right)\frac{1}{2}}$$

$$= \frac{1}{i}\oint \frac{dz}{z} \cdot \frac{1}{b+\frac{z^2+1}{2z}} = \frac{1}{i}\oint \frac{1}{bz+\frac{z^2+1}{2}} = \frac{1}{i}\oint \frac{2}{2bz+z^2+1} = J$$

(b) Use Cauchy's Residue Calculus to evaluate J. **Explain why** $b > 1$.

$z^2+2bz+1 = 0 \Rightarrow z = \frac{-2b \pm \sqrt{(2b)^2-4\cdot1\cdot1}}{2} = -b \pm \sqrt{b^2-1}$

Since $b>1$, $z^* = -b+\sqrt{b^2-1}$ will be inside $|z|<1$ and $-b-\sqrt{b^2-1}$ will not

$$\text{Res}\left(-b+\sqrt{b^2-1}, \frac{2}{z^2+2bz+1}\right) = \lim_{z\to z^*} z-z^* \cdot \frac{2}{z^2+2bz+1} = \left.\frac{2}{2z+2b}\right|_{z=z^*}$$

$$= \frac{1}{z^*+b} = \frac{1}{-b+\sqrt{b^2-1}+b} = \frac{1}{\sqrt{b^2-1}} \quad \text{So } J = 2\pi i \cdot \frac{1}{i}\frac{1}{\sqrt{b^2-1}}$$

$$= \frac{2\pi}{\sqrt{b^2-1}}$$

$b>1$ so I is NOT improper
so $b+\cos\theta > 0$ for all $0 \le \theta \le 2\pi$.

MATH PART II:

(10 points) The goal of this problem is to demonstrate the properties of the **Laguerre polynomials** $L_n(x)$, that satisfy the Laguerre differential equation: $xy''+(1-x)y'+ny=0$.

(a) Use the Rodrigues' formula for the Laguerre polynomials, $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$, to confirm that $L_0(x) = 1$, $L_1(x) = 1 - x$, and to find $L_2(x)$.

$n=0$

$L_0 = \frac{e^x}{0!} \cdot x^0 \cdot e^{-x} = 1$

$n=1$

$L_1 = \frac{e^x}{1!}\frac{d}{dx}(x^1 e^{-x})$

$= e^x(1 \cdot e^{-x} - x \cdot e^{-x})$

$= 1 - x$

$n=2$

$L_2 = \frac{e^x}{2!}\frac{d^2}{dx^2}(x^2 e^{-x})$

$= \frac{e^x}{2}\frac{d}{dx}\left[2xe^{-x} - x^2 e^{-x}\right]$

$= \frac{e^x}{2}\left[2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2 e^{-x}\right]$

$= 1 - x - x + \frac{x^2}{2} = 1 - 2x + \frac{x^2}{2}$

(b) To confirm your expression for $L_2(x)$ found in part (a) is correct, use one of the recurrence relations $\mathbf{(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)}$ ~~or $L'_n(x) = L'_{n-1}(x) - L_{n-1}(x)$~~.

$n=1$

$2L_2(x) = (3-x)L_1(x) - 1 \cdot L_0(x)$

$= (3-x)(1-x) - 1$

$= 3 - x - 3x + x^2 - 1$

$= 2 - 4x + x^2 \Rightarrow L_2(x) = 1 - 2x + \frac{x^2}{2}$ ✓

(c) Verify that your expressions for $L_0(x)$, $L_1(x)$, and $L_2(x)$ satisfy the Laguerre differential equation $xy''+(1-x)y'+ny=0$.

$L_0' = 0 \quad L_0'' = 0$

$n=0$ LHS $= xL_0'' + (1-x)L_0' + 0 \cdot L_0$

$= 0 + 0 + 0 =$ RHS

$L_1' = -1 \quad L_1'' = 0$

$n=1$

LHS $= xL_1'' + (1-x)L_1' + 1 \cdot L_1$

$= x \cdot 0 + (1-x)(-1) + 1 - x$

$= 0 + x - 1 + 1 - x$

$= 0 =$ RHS

$L_2' = -2 + x$

$L_2'' = 1$

$n=2$

LHS $= xL_2'' + (1-x)L_2' + 2 \cdot L_2$

$= x \cdot 1 + (1-x)(x-2) + 2\left(1 - 2x + \frac{x^2}{2}\right)$

$= x + x - 2 - x^2 + 2x + 2 - 4x + x^2 = 0 =$ RHS

HISTORY (______ of 25 POINTS)

HISTORY PART I: LONG-ANSWER QUESTION (10 points).
If Euler or Gauss had never been born, whom do you think would be considered the most important (i.e. historically significant) mathematician of the 18th and 19th centuries?
WRITE LEGIBLY and provide 1-3 paragraphs (i.e. each with multiple sentences) to support your answer! Be sure to include biographical ad mathematical information about the person you select.

(I'd pick Cauchy)

Cauchy made incredible contributions to multiple areas of mathematics, basically creating complex analysis. Cauchy-Riemann equations, Cauchy Integral Formula, Cauchy Residue Calculus, etc are just some of the concepts named after him.

Cauchy published more than 900 papers in his lifetime and had an outsized influence on mathematics. He had leadership roles in important institutions. ~~too~~

Cauchy was a proponent of increasing rigor in mathematics. He popularized and explained Calculus to thousands through the publication of his lecture notes (titled <u>Cour's d'analyse</u>).

HISTORY PART II: MATCH QUESTION (5 points)
Match the concept, symbol or equation with the name of the one Mathematician most closely associated with it. (You can write down the number next to the letter.)

A: $P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),\ \ldots$

B: The continuum hypothesis

C: $e^{\pi i}+1=0$

D: The fundamental theorem of algebra

E: A function which is nowhere continuous

1. **Leonhard Euler**
2. **Carl Friedrich Gauss**
3. **Joseph-Louis Lagrange**
4. **Pierre-Simon Laplace**
5. **Adrien-Marie Legendre**
6. **Augustin-Louis Cauchy**
7. **Joseph Fourier**
8. **Johann Dirichlet**
9. **Evariste Galois**
10. **Georg Cantor**

A 5 **B** 10 **C** 1 **D** 2 **E** 8

HISTORY PART III: SHORT-ANSWER QUESTIONS (10 points)
Write down whether the following sentences are either TRUE or FALSE (by writing a T or F on the blank line. USE THE SPACE BETWEEN POINTS TO EXPLAIN YOUR ANSWER.

A. F Kronecker, Poincaré and Cantor were all proponents of the idea that multiple sizes of infinity exist.

Kronecker and Poincaré both HATED Cantor's theories about infinity.

B. T Gauss and Cauchy both have probability density functions named after them.

Yes, the Gaussian is well-known but Cauchy also has a pdf as described in a class worksheet.

C. F Euler is known as the "Prince of Mathematicians."

Gauss, not Euler, is known as the "Prince of Math." Euler is called "Master of us all" (by Laplace)

D. F Weierstrass and Abel were on opposite sides of the effort to increase mathematical rigor in the nineteenth century.

Nope! They both were in favor of increased rigor.

E. T The Basel Problem and the Konigsberg Bridge Problem were solved by Galois.

Yes both of these were solved by Euler.