History of Mathematics

Math 395 Spring 2010 — Fowler 310 MWF 10:30am - 11:25am
 — http://faculty.oxy.edu/ron/math/395/10/

Class 31: **Friday April 23**

TITLE Twentieth Century Mathematics and Mathematicians
CURRENT READING: Katz, §25
NEXT READING: Katz, §25

Homework #10 on Friday April 23rd
Katz, page 636-639: 2,17, 18, 39. EXTRA CREDIT: 32, 38.

SUMMARY
We will discuss some of the advances in mathematics in the 1900 and 2000s.

David Hilbert (1862-1943)
Hilbert is most well-known for his address on August 8,1900 to the 2nd International Congress of Mathematicians in Paris where he presented important and unsolved problems in mathematics. His name is prominent in the field of functional analysis, where a Hilbert space is an important concept.

The 23 Problems
David Joyce has posted the full text of Hilbert's article in which the original statement of the 23 problems appears at: http://aleph0.clarku.edu/~djoyce/hilbert/problems.html

1. Cantor's problem of the cardinal number of the continuum.
2. The compatibility of the arithmetical axioms.
3. Give two tetrahedra that cannot be decomposed into congruent tetrahedra directly or by adjoining congruent tetrahedra.
4. Problem of the straight line as the shortest distance between two points.
5. Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group
6. Mathematical treatment of the axioms of physics
7. Irrationality and transcendence of certain numbers
8. **Problems of prime numbers**
9. Proof of the most general law of reciprocity in any number field
10. Determination of the solvability of a diophantine equation
11. Quadratic forms with any algebraic numerical coefficients
12. **Extension of Kroneker's theorem on Abelian fields to any algebraic realm of rationality**
13. Impossibility of the solution of the general equation of the 7-th degree by means of functions of only two arguments
14. Proof of the finiteness of certain complete systems of functions
15. Rigorous foundation of Schubert's enumerative calculus
16. Problem of the topology of algebraic curves and surfaces
17. Expression of definite forms by squares
18. Building up of space from congruent polyhedra
19. Are the solutions of regular problems in the calculus of variations always necessarily analytic?
20. The general problem of boundary values
21. Proof of the existence of linear differential equations having a prescribed monodromic group
22. Uniformization of analytic relations by means of automorphic functions
23. Further development of the methods of the calculus of variations

Twentieth Century Mathematics
It should be noted that most of the problems of Hilbert stated that could be resolved have been resolved, with the mathematical community considering 3, 7, 10, 11, 13, 14, 17, 19, 20, and 21 solved and 4, 6, 16 and 23 too vague to have a solution with only 8 and 12 being completely unresolved. Many people feel that 1, 2, 5, 9, 15, 18 and 22 have solutions there is not consensus on these.

It should also be noted that there has been more original mathematics produced in the twentieth century than that of all the rest of the previous centuries combined, but that almost all of the mathematics one learns as an undergraduate was basically settled before Hilbert's address.

Problem #8: The Riemann Hypothesis
The most famous of these problems involves the zeroes of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{(1 - \frac{1}{p^s})}$$

The result is an amazing one of Euler's, who also showed that

$$\zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^4}{90}$$

And Euler went on to produce a general formula for $\zeta(2n)$. Riemann, however, went even further and showed that by making s a complex number and expressing the function in terms of contour integrals he discovered that $\zeta(-2n)=0$. He also mentioned in passing that it was "very likely" that all the complex zeroes of the zeta function had real part equal to ½ but could not prove it. It should be noted that $\zeta(s)$ can be related to $\pi(x)$, the number of primes less than x, which is why it is important. To date, computer calculations have shown that the first 1,500,000,000 zeroes of the Riemann zeta function do indeed satisfy the "Riemann hypothesis" but this is not considered a proof. What do you think?

Problem #1: The continuum hypothesis
The continuum hypothesis is about the nature of infinity. In other words, is there a set of numbers which has a cardinality between the set of real numbers and the set of natural numbers?

Kurt Gödel (1906-1978) proved that it was impossible to disprove the continuum hypothesis assuming that the standard axioms of set theory are true. Gödel is well-known for his incompleteness theorems. The first basically states that no consistent system of standard axioms which can be listed by a computer is able to prove all possible facts about the natural numbers and the second says that if a system of axioms is capable of proving basic facts about the natural numbers than the one thing it can not prove is whther the set of axioms itself is consistent. By putting these theorems together Gödel (at age 25) was widely considered to have solved Hilbert's 2nd problem with a definitive "no!"

George Cantor (1845-1918)
Widely regarded as the father of transfinite set theory, Cantor showed that the set of natural numbers and the set of real numbers have different cardinalities. The cardinality of the natural numbers he described as $\aleph_0$ (aleph-zero) which the cardinality of the real numbers is denoted **c**.

GroupWork

What are the cardinalities of the following sets?

$\mathbb{R}$ (real numbers), $\mathbb{Z}$ (integers), $\mathbb{N}$ (natural numbers), $\mathbb{Q}$ (rational numbers)

What The Heck Are the Real Numbers?

Richard Dedekind (1831-1916) and **Karl Weierstrass (1815-1897)** proved some of the seminal results in real analysis in the 19th century. One of the biggest problems was coming up with a formal definition of the real numbers.

Dedekind cuts

One way to construct the real numbers was to use an object called a Dedekind cut. Dedekind defined a *schnitt* (German word for cut) as a number α such that it breaks the set of real numbers R into two classes A_1 and A_2 whereby every number α_1 in A_1 is less than every number α_2 in A_2. He proved that there existed one unique number α which is either the greatest number in A_1 or the smallest number in A_2. In fact, he defined α as the rational number corresponding to the cut (A_1, A_2). He then also proved that the system R has the properties of continuity and natural ordering.

Other ways of defining the real numbers involve Cauchy sequences.

Twenty-First Century Mathematics

In May 2000, the Clay Mathematics Institute, following in the footsteps of David Hilbert a century earlier, announced the Millenium Prize of $1,000,000 for the successful solution of any of the following seven open problems in mathematics. Learning from Hilbert's mistake, the problems were very clearly stated

1. Birch and Swinnerton-Dyer Conjecture. (It is related to Hilbert's 10th problem and involves whether certain Diophantine equations have rational solutions)
2. Hodge Conjecture. (Involves algebraic geometry and algebraic topology)
3. Poincaré Conjecture (**Solved! Prize awarded on March 10, 2010 to Grigory Perelman**) It is a topology problem which is about the classification of 3-manifolds and their relation to the sphere. Perelman discovered his proof in 2002 and in 2006 it was verified and he was awarded the Fields Medal, the highest honor in mathematics (which he refused).
4. Riemann Hypothesis.
5. P versus NP. (Can a computer that verify a problem is solved in polynomial time also solve such a problem as fast?)
6. Yang-Mills Existence and Mass gap theory. (Prove that the smallest possible particle in quantum field theory must have positive mass.)
7. Navier-Stokes existence and uniqueness. (This is basically about proving the equation of fluid dynamics always have smooth solutions and an attempt to describe the nature of turbulence.)

Andrew Wiles (1953-)

Wiles is a Princeton math professor who finally proved that Fermat's Last Theorem (there are no integers such that the equation $x^n + y^n = z^n$ is true for $n>3$) in 1995.

Exercise

Name another 20th or 21st century mathematician!