History of Mathematics

Class 30: **Wednesday April 21**

TITLE Cauchy: The Perfectionist
CURRENT READING: Katz, §22
NEXT READING: Katz, §25

Homework #10 on Friday April 23rd
Katz, page 636-639: 2,17, 18, 39. EXTRA CREDIT: 32, 38.

SUMMARY
We will discuss the dramatic impact Cauchy had on strengthening the rigor of Calculus.

Augustin-Louis Cauchy (1789-1857)
Cauchy was one of the most prolific mathematicians of all-time, probably second behind Euler. Cauchy was born in 1789 and initially educated by his father, although he began attending *Ecole Polytechnique* in 1805, where he gained attention from Lagrange and Laplace after placing second out of nearly 300 in the entrance exam. In 1807, Cauchy went on to attend *École des Ponts et Chaussées* (Schools of Bridges and Highways) to become a civil engineer. However his engineering career was short-lived and by 1816 Cauchy was a full professor of mathematics at *École Polytechnique*

Analytical Rigor
Cauchy published his most famous work, *Cours d'analyse de l'École Royale Polytechnique* (Analysis Course from the Royal Polytechnic School) in 1821. In it, he presents a rigorous approach to some of the central concepts in Calculus: limit, continuity, function, derivative and integral. He then used these definitions to improve some of the most important theorems in Calculus. Additionally, he brought order to the use of infinite series by devising tests which could determine their convergence (Cauchy Integral Test, Cauchy Root Test, etc).

GroupWork
Give the formal modern definition of **function, limit of a function, continuity** and **derivative**.

Katz compiles some definitions of these terms throughout time in sidebars on page 768, 770 and 783.

Definitions of "limit"

Leibniz (1684): If any continuous transition is proposed terminating in a certain limit, then it is possible to form a general reasoning, which covers also the final limit.

Newton (1687): The ultimate ratio of evanescent quantities . . . [are] limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*.

Maclaurin (1742): The ratio of $2x + o$ to a continually decreases while o decreases and is always greater than the ratio of $2x$ to a while o is any real increment, but it is manifest that it continually approaches to the ratio of $2x$ to a as its limit.

D'Alembert (1754): This ratio $[a : 2y + z]$ is always smaller than $a : 2y$, but the smaller z is, the greater the ratio will be and, since one may choose z as small as one pleases, the ratio $a : 2y + z$ can be brought as close to the ratio $a : 2y$ as we like. Consequently, $a : 2y$ is the limit of the ratio $a : 2y + z$.

Lacroix (1806): The limit of the ratio $(u_1 - u)/h$ . . . is the value towards which this ratio tends in proportion as the quantity h diminishes, and to which it may approach as near as we choose to make it.

Cauchy (1821): If the successive values attributed to the same variable approach indefinitely a fixed value, such that they finally differ from it by as little as one wishes, this latter is called the limit of all the others.

Definitions of "continuity"

Euler (1748): A continuous curve is one such that its nature can be expressed by a single function of x. If a curve is of such a nature that for its various parts . . . different functions of x are required for its expression, . . . , then we call such a curve discontinuous.

Bolzano (1817): A function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits . . . if [when] x is some such value, the difference $f(x + \omega) - f(x)$ can be made smaller than any given quantity provided ω can be taken as small as we please.

Cauchy (1821): The function $f(x)$ will be, between two assigned values of the variable x, a continuous function of this variable if for each value of x between these limits, the [absolute] value of the difference $f(x + \alpha) - f(x)$ decreases indefinitely with α.

Dirichlet (1837): One thinks of a and b as two fixed value: and of x as a variable quantity that can progressively take al values lying between a and b. Now if to every x there corre sponds a single, finite y in such a way that, as x continuousl passes through the interval from a to b, $y = f(x)$ also gradu ally changes, then y is called a continuous function of x in thi interval.

Heine (1872): A function $f(x)$ is continuous at the particula value $x = X$ if for every given quantity ϵ, however small, ther exists a positive number η_0 with the property that for no positiv quantity η which is smaller than η_0 does the absolute value o $f(X \pm \eta) - f(X)$ exceed ϵ. A function $f(x)$ is continuou from $x = a$ to $x = b$ if for every single value $x = X$ betwee $x = a$ and $x = b$, including $x = a$ and $x = b$, it is continuous

Definitions of "function"

Johann Bernoulli (1718): I call a function of a variable magnitude a quantity composed in any manner whatsoever from this variable magnitude and from constants.

Euler (1748): A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.

Euler (1755): When quantities depend on others in such a way that [the former] undergo changes themselves when [the latter] change, then [the former] are called functions of [the latter]; this is a very comprehensive idea which includes in itself all the ways in which one quantity can be determined by others.

Lacroix (1810): Every quantity whose value depends on one or more other quantities is called a function of these latter, whether one knows or is ignorant of what operations it is necessary to use to arrive from the latter to the first.

Fourier (1822): In general, the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there is an equal number of ordinates $f(x)$. All have actual numerical values, either positive or negative or null. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity.

Heine (1872): A single-valued function of a variable x is an expression which for every single rational or irrational value of x is uniquely defined.

Dedekind (1888): A function ϕ on a set S is a law according to which to every determinate element s of S there belongs a determinate thing which is called the transform of s and denoted by $\phi(s)$.

Cauchy's definition of the integral

Suppose f(x) is continuous on a closed interval [x_0,X] and there are n-1 values in this interval such that $x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = X$ then the sum S given by

$$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \ldots + (X - x_{n-1})f(x_{n-1})$$

Which can also be re-written

$$\sum_{k=1}^{n}(x_k - x_{k-1})f(x_{k-1})$$

Cauchy showed that the partition (the choice of values in the interval) were irrelevant to the final value of S as long as n gets larger and larger, and he called the limitof this process the definite integral which is written

$$\int_{x_0}^{X} f(x)dx$$

Mean Value Theorem for Integrals

For some $0 \leq \theta < 1$

$$\int_{x_0}^{X} f(x)dx = (X - x_0)f[x_0 + \theta(X - x_0)]$$

Fundamental Theorem of Calculus

Given *f(x)* is continuous in [x_0,X] and x is in [x_0,X] and $F(X) = \int_{x_0}^{X} f(x)dx$, then ***F'(X)=f(x).***

EXAMPLE

Let's prove the fundamental theorem of Calculus the way Cauchy did, using the mean value theorem and the additive property for definite integrals by using the limit definition of the derivative (also derived by Cauchy) on the function F(X).

Cauchy and Complex Analysis
Cauchy contributed greatly to the analysis of a function of a complex variable and there are numerous results in the field which bear his name

George Friedrich Bernhard Riemann (1826-1866) was a German mathematician who contributed greatly to analysis contemporaneously with Cauchy. He is most well-known for his contributions to differential geometry and calculus.

Cauchy-Riemann Equations
Given $Z(x+iy) = M(x,y) + i\,N(x,y)$ the Cauchy-Riemann equations are

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x} \qquad \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

It turns out that the CRE are satisfied if the function $Z(x+iy)$ is analytic. Analyticity is a very important property of functions of a complex variable. It means that it can be expanded into a convergent infinite series for all points z in an open disk.

Cauchy Integral Theorem
If the function $f(z)$ is analytic, the value of the integral in the complex plane is independent of the path taken, and is zero on a closed path.

$$\oint_\gamma f(z)\,dz = 0.$$

Cauchy Integral Formula
Given an analytic function $f(z)$

$$f(a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\,dz$$

Cauchy Differentiation Formula
The CIF leads immediately to the following result

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

Cauchy's Error
Cauchy falsely proved that an infinite series of continuous functions would be continuous. Abel found the error, which was an example of a Fourier series. Abel's counter-example was:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$$

Which represents the periodic function $x/2$ on $[0,\pi/2)$ with period π.

Jean Baptiste Joseph Fourier (1768-1830)
Fourier is most well-known for his assertion (strongly resisted by Lagrange and Laplace) that any function could be represented by an infinite series of sines or cosines in his seminal work *Théorie analytique de la chaleur* (Analytical Theory of Heat).

Niels Henrik Abel (1802-1829)
Abel's life was brief but he made an indelible mark upon the history of mathematics. At age 16 he wrote a paper proving the exponent in the binomial expansion was valid for all real numbers. Several papers were published after his death which led to the development of group theory