History of Mathematics

Math 395 Spring 2010 Fowler 310 MWF 10:30am - 11:25am
 http://faculty.oxy.edu/ron/math/395/10/

Class 28: **Wednesday April 14**

TITLE Euler: The Master Of Us All
CURRENT READING: Katz, §17.3, §16.3
NEXT READING: Katz, §17.1-17.2

Quiz #3 on Friday April 16th
On Chapters 12-17 with an emphasis on the 17th and 18th centuries.

SUMMARY
We will concentrate on the work of Euler, who Laplace called "the master of us all."

Boyer's *A History of Mathematics* says about Euler's 1748 work *Introductio in analysin infinitorum (Introduction to Analysis of the Infinite)*:

> **It may be fairly said that Euler did for the infinite analysis of Newton and Leibniz what Euclid had done for the geometry of Eudoxus and Theaetetus, or what Viete had done for the algebra of al-Khowarizmi and Cardan. Euler took the differential calculus and the method of fluxions and made them part of a more general branch of mathematics which has been known as "analysis"--the study of infinite processes. If the ancient *Elements* was the cornerstone of geometry and the medieval *Al jabr wa'l muqbalah* was the foundation stone of algebra, then Euler's *Introductio in analysin infinitorum* can be thought of as the keystone of analysis.**

Calculus Results
Euler also published *Institutiones calculi differentialis (Method of the Differential Calculus)* in 1755 and *Institutiones calculi integralis (Method of the Integral calculus)* in 1768 .

Euler used infinite series in clever ways to prove the formula for the quotient rule and derivative of the logarithm.

First he expanded

$$\frac{1}{q+dq} = \frac{1}{q}\left(1 - \frac{dq}{q} + \frac{(dq)^2}{q^2} - \dots\right)$$

And neglecting higher order terms leads to

$$\frac{p+dp}{q+dq} = (p+dp)\left(\frac{1}{q} - \frac{dq}{q^2}\right)$$

Which allows you to write an expression for $d\left(\frac{p}{q}\right)$. Similarly, Euler used

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ to produce an expression for the derivative of the logarithm

Example

Show that $d\left(\frac{p}{q}\right) = \frac{qdp - pdq}{q^2}$ and when $y = \ln x$, $dy = \frac{dx}{x}$

Complex Numbers

Euler is very well known for the result that

$$e^{ix} = \cos(x) + i\sin(x)$$

Which of course when $x=\pi$ leads to "the most beautiful equation in all of mathematics":

$$e^{i\pi} + 1 = 0$$

What's interesting about the above result that is Euler himself popularized our common understanding of the symbols e and π which appear in it.

Euler was also able to show that

$$i^i = e^{-\pi/2}$$

Which is another amazing result.

Exercise

Let's try to show that $i^i = e^{-\pi/2}$. Use the fact

$$z = a + bi = re^{i\theta} \text{ and } z^c = e^{\log(z^c)} = e^{c\log(z)}$$

The Many Sides of Euler

Euler has his name on so many concepts, procedures and ideas that it has become customary to name things for the person who rediscovered them first ***after*** Euler.

Here are just some of the classic results which have survived to keep his name.

Euler's Constant

A constant which shows up in Number Theory which measures the difference between the sum of the harmonic series and the natural logarithm function as they both diverge.

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right) = 0.57721...$$

Euler's Method

This is a method for approximating solutions to differential equations of the form

$$\frac{dy}{dt} = f(y,t), \quad y(a) = b$$

Numerically using the algorithm

$$y_{n+1} \approx y_n + f(y_n, t_n)(t_{n+1} - t_n)$$

Euler characteristic

Stemming from his solution of the famous Seven Bridges of Konigsberg problem, Euler showed that a finite, connected planar graph (with no edge intersections) has the property that there was a simple relationship between the number of **vertices V, edges E** and **faces F** such that **V-E+F=2**.

Euler-Lagrange Equation

The Euler-Lagrange equation is the basic equation of the area of mathematics called the calculus of variations. Given an integral of the form $I(y) = \int_a^b F(x, y, y')dx$ the problem is to find a curve *y(x)* hich minimizes or maximizes the value of I. The condition that *y(x)* must satisfy is called the Euler-Lagrange equation and looks like

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Euler Differential Equation

An Euler (ordinary differential) equation is one of the form

$$a_2x^2y''(x) + a_1xy'(x) + a_0y = f(x)$$

Where the order of the derivative and the order of the independent variable are paired. Euler showed that using the transformation $x = e^t$ one can transform this into a constant coefficient differential equation, which is then very easy to solve.

GroupWork

A. Draw your own planar graph and compute the number of vertices, edges and faces.

B. Convert the ordinary differential equation $x^2 y''+5xy'+6y = 0$ and convert into a constant coefficient ODE and find the general solution.

C. Use the method of Calculus of Variations to prove that the shortest distance between two points is a straight line. In other words, considering $F(x, y, y') = \sqrt{1 + y'^2}$ solve the Euler-Lagrange equation.