History of Mathematics

Math 395 Spring 2010 Fowler 310 MWF 10:30am - 11:25am
 http://faculty.oxy.edu/ron/math/395/10/

Class 27: **Monday April 12**

TITLE 18th Century Greats: Clairault, D'Alembert, Euler, Lagrange, Laplace and Legendre
CURRENT READING: Katz, §17.3, §16.3
NEXT READING: Katz, §17.1-17.2

Quiz #3 on Friday April 16th
On Chapters 12-17 with an emphasis on the 17th and 18th centuries.

SUMMARY
We will discuss some of the work of the greats who defined mathematics of the late 18th century.

By the end of the 18th century mathematicians were beginning to feel that mathematics was "exhausted" (Struik, 136). Francois Arago wrote in 1842's *Éloge de Laplace* (Eulogy for Laplace):

> **Five geometers—Clairault, Euler, d'Alembert, Lagrange, and Laplace—shared between them the world of which Newton had revealed the existence. They explored it in all directions, penetrated into regions believed inaccessible, pointed out countless phenomena in those regions which observation had not yet detected, and finally—and herein lies their imperishable glory—they brought within the domain of a single principle, a unique law, all that is most subtle and mysterious in the motions of the celestial bodies.**

Alexis-Claude Clairault (1713-1765)
Clairault was born and died in Paris. He is most well-known for his work in differential equations. He was a child prodigy who published his first paper at age 12 and was elected to the Paris Academy of Sciences at age 18. He is also known for going on an expedition which confirmed a result of Newton and Huygens that the Earth would be flattened (less spherical) at the poles that was published in his *Théorie de la figure de la Terre* (A theory on the shape of the Earth).

Joseph Louis Lagrange (1736-1813)
Lagrange was born in Turin, Italy and Boyer ranks him as the second greatest mathematician of the century (behind Euler). When Euler left Berlin for St. Petersburg, Lagrange took the position the Swiss great had vacated. Lagrange is most well-known for trying to come up with a more analytically rigorous foundation for the calculus, and in 1797 he published *Théorie des fonctions analytique contenant les principes du calcul differential* (*Theory of analytic functions containing the principles of differential calculus*). In his masterwork of 1787, *Mécanique analytique* (*Analytic Mechanics*) he gave the general equations of motion of a dynamical system which are today known as Lagrange's Equations. In the preface of this book Lagrange wrote "*On ne trouvera point de figures dans cet ouvrage, seulement des opérations algébriques" (One will not find figures in this work, only algebraic operations)*. This was done to dramatically separate this work from the classic geometric texts of the Greeks and confirm the modern supremacy of algebra and analysis.

Leonhard Euler (1707-1783)

Euler was born in Basel, Switzerland but spent most of his life in St. Petersburg, Russia and Berlin, Germany. He is widely regarded as the most prolific mathematician of all time, with significant contributions to various branches of mathematics. He is most well-known for his establishment of particular symbols which his use led to their universal adoption. Boyer presents a small list of these symbols we use now that are attributable to Euler:

$f(x)$	For functional notation
e	For the base of the natural logarithms
a, b, c	For the sides of the triangle ABC
s	For the semiperimeter of triangle ABC
r	For the inradius of the triangle ABC
Σ	For the summation sign
i	For the imaginary unit $\sqrt{-1}$

Jean le Rond D'Alembert (1717-1783)

D'Alembert was named for the church, Saint-Jean-le-Rond-de-Paris, he was found outside of as an infant, abandoned by his mother due to his illegitimate status. D'Alembert was a rival of Clairault's and was admitted to the French Academy of Sciences at the ripe old age of 24. D'Alembert is most well-known for his proof of the Fundamental Theorem of Algebra (which in France is known as the d'Alembert-Gauss Theorem) as well as his 1743 book *Traite de dynamique* (Treatise on dynamics) in which he derives and proposes a solution for one of the three fundamental partial differential equations of mathematical physics, the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$. D'Alembert became the permanent secretary of the French Academy and contributed greatly to the publication of *Encyclopédie*, the huge encyclopedic publication of the French Academy which attempted to compile all the knowledge of the Enlightenment.

Pierre-Simon Laplace (1749-1827)

Laplace was a contemporary of the others but didn't publish until the turn of the 19^{th} century. His *Traité de mécanique céleste* (*Treatise on Celestial Mechanics*) earned him the sobriquet "the Newton of France" and was a 5-volume compendium of all discoveries in this field to date. Laplace is often quoted and there are multiple amusing anecdotes attached to his name. For example, when Napoleon Bonaparte mentioned that God is not mentioned in the *Treatise*, Laplace replied "*Sire, je n'avais pas besoin de cette hypothèse*." ("Sire, I had no need of that hypothesis.") Laplace's name is associated with one of the other fundamental partial differential equations, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ as well as an integral transform which is used to solve initial value problems with piecewise-continuous functions. Laplace was often compared (unfavorably) with Lagrange. Laplace rarely gave proofs of his results and seemed uninterested in rigor while, Lagrange can be considered the first true analyst (Boyer).

Adrien-Marie Legendre (1752-1833)

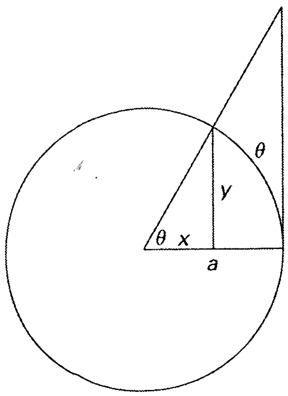
Legendre was a contemporary of Laplace and Lagrange but is often not as well-known as these other two famous mathematicians. Legendre's name is most closely associated with the differential equation $(1-x^2)y''-2xy'+n(n+1)y = 0$, and particular the functions which satisfy this equation, which when n is a nonnegative integer are called Legendre polynomials. His *Éléments de géometrie* of 1794 was a significant pedagogical improvement of Euclid's *Elements* and became the most popular geometry textbook for over a century in America and England after its translation in 1819.

Euler and D'Alembert's Argument Over Logarithm of Negative Numbers
D'Alembert (and other mathematicians like John Bernouilli) were convinced that log(-*x*)=log(x) because

$$(-x)^2 = x^2$$
$$\log[(-x)^2] = \log[x^2]$$
$$2\log(-x) = 2\log(x)$$
$$\log(-x) = \log(x)$$

However, Euler in a letter entitled ON THE CONTROVERSYBETWEEN Messrs. LEIBNIZ AND BERNOULLI CONCERNING LOGARITHMS OF NEGATIVE AND IMAGINARY NUMBERS to D'Alembert which was later published explained how to compute the logarithm of any negative or complex number.

Consider page 597 of Katz



Given $x=\cos\theta$ and $y=\sin\theta$ then $\theta=\arctan(y/x)=\arctan(t/a)$, so

$$\frac{a^2}{2}\theta = \frac{a^2}{2}\arctan\frac{t}{a}$$
$$= \frac{a^2}{2}\int\frac{a\,dt}{a^2+t^2} = \frac{a^2}{2}\int\left[\frac{1/2}{a+ti}+\frac{1/2}{a-ti}\right]dt$$
$$= \frac{a^2}{4}\left[\frac{1}{i}\ln(a+ti) - \frac{1}{i}\ln(a-ti)\right] = \frac{a^2}{4i}\ln\left[\frac{a+ti}{a-ti}\right]$$
$$= \frac{a^2}{4i}\ln\left[\frac{a+(ay/x)i}{a-(ay/x)i}\right] = \frac{a^2}{4i}\ln\left[\frac{x+yi}{x-yi}\right]$$

Let $x=0$ and $\theta=\pi/2$ gives you the result that

$$\frac{\pi a^2}{4} = \frac{a^2}{4i}\ln(-1)$$

Or $\ln(-1)=i\pi$.

In *Complex Analysis* we learn that

$$\log z = \log(x+iy) = \ln\left(\sqrt{x^2+y^2}\right) + i(\theta+2n\pi)$$

where $\theta=\arctan(y/x)$.

GroupWork

Compute log(-4), log(-3+4i), log(i) and log(0).

Sum of Infinite Series

Leibniz, Jakob Bernouilli and many other mathematicians were stumped by the problem of coming up with a closed form of the sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

However, Euler was able to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$$

Euler used two ideas,

1) The sum of the reciprocals of the roots of a polynomial equation written as $1 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n$ is equal to the negative coefficient of the linear term, i.e. $-c_1$. (This was a well-known result first proved by Descartes) and

2) the "Maclaurin" series expansion of $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ..$

Eves explains:

> Then $\sin z = 0$ can (after dividing through by z) be considered as the infinite polynomial
>
> $$1 - z^2/3! + z^4/5! - z^6/7! + \cdots = 0,$$
>
> or, replacing z^2 by w, as the equation
>
> $$1 - w/3! + w^2/5! - w^3/7! + \cdots = 0.$$
>
> By the theory of equations, the sum of the reciprocals of the roots of this equation is the negative of the coefficient of the linear term—namely, $\frac{1}{6}$. Since the roots of the polynomial in z are $\pi, 2\pi, 3\pi, \ldots$, it follows that the roots of the polynomial in w are $\pi^2, (2\pi)^2, (3\pi)^2, \ldots$. Therefore
>
> $$\tfrac{1}{6} = 1/\pi^2 + 1/(2\pi)^2 + 1/(3\pi)^2 + \cdots,$$
>
> or
>
> $$\pi^2/6 = 1/1^2 + 1/2^2 + 1/3^2 + \cdots.$$

Exercise

Use Euler's Method and the Maclaurin expansion for $\cos z$ to show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$