History of Mathematics

Math 395 Spring 2010
Fowler 310 MWF 10:30am - 11:25am
http://faculty.oxy.edu/ron/math/395/10/

Class 24: **Monday April 5**

TITLE The Calculus: Gottfried Wilhelm Leibniz
CURRENT READING: Katz, §16.2
NEXT READING: Katz, §16.3

Homework #9 DUE Friday April 9 (in class)
Katz, p. 539: #1, #6, #11, #25, #26. EXTRA CREDIT: page 540, #18, #19.

SUMMARY
We will consider the life and work of the other inventor of the Calculus: Gottfried Leibniz.

Gottfried Wilhelm Leibniz (1646-1716) is often called a polymath (i.e. universal genius) of the first order for his intellectual contributions to the fields of mathematics, philosophy, physics, theology, ethics and logic. He is widely regarded as a second, independent developer of differential and integral calculus as well as the primary inventor of the modern binary number system.

Leibniz was very conscious of the significance of notation and naming. It is his symbols which have mostly persisted in modern representations of Calculus, such as *dx* and $\int$. Leibniz also coined the terms differential calculus and integral calculus.

The Harmonic Triangle
Leibniz was mainly self-taught in mathematics and thus often rediscovered on his own previously known results. He began his foray into discovering Calculus by noticing patterns in the sums of differences, particularly in Pascal's Triangle and in a new structure he called "the harmonic triangle."

Arithmetic triangle

$$\begin{array}{lllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1\cdots \\
1 & 2 & 3 & 4 & 5 & 6\cdots \\
1 & 3 & 6 & 10 & 15\cdots \\
1 & 4 & 10 & 20\cdots \\
1 & 5 & 15\cdots \\
1 & 6\cdots \\
1\cdots
\end{array}$$

Harmonic triangle

$$\begin{array}{llllll}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6}\cdots \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \frac{1}{30}\cdots \\
\frac{1}{3} & \frac{1}{12} & \frac{1}{30} & \frac{1}{60}\cdots \\
\frac{1}{4} & \frac{1}{20} & \frac{1}{60}\cdots \\
\frac{1}{5} & \frac{1}{30}\cdots \\
\frac{1}{6}\cdots
\end{array}$$

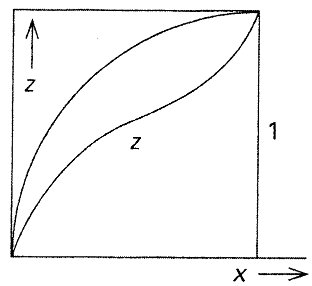
In the arithmetic triangle each element (which is not in the first column) is the difference of the two terms directly below it and to the left; in the harmonic triangle each term (which is not in the first row) is the difference of the two terms directly above it and to the right. Moreover, in the arithmetic triangle each element (not in the first row or column) is the sum of all of the terms in the line above it and to the left, whereas in the harmonic triangle each element is the sum of all of the terms in the line below it and to the right.

Leibniz was able to realize this discrete situation could be replicated in the continuous context of an infinite number of ordinates representing points on a curve.

$$\sum_{i=1}^{n} \delta y_i = y_n - y_0 \qquad \int dy = y$$

$$\delta \sum_{i=1}^{n} y_i = y_n - y_0 \qquad d \int y = y$$

Leibniz' Quadrature of the Circle



The area of a quarter of a unit circle can be represented by the following integral

$$\int y\ dx = 1 - \int \frac{z^2}{1+z^2} dz$$

But Leibniz (similar to Newton) was able to use infinite series as tools and thus approximated $(1+z^2)^{-1}$ to obtain his first famous result

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

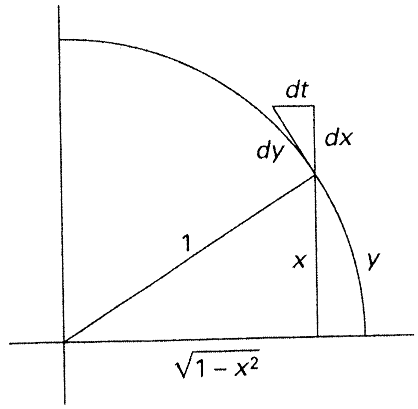
Derivative Rules

In *"A New Method for Maxima and Minima as well as Tangents, which is neither impeded by fractional or irrational Quantities and a Remarkable type of Calculus for them"* of 1684 showed how to work with differentials, which produces rules that look very much like our modern Calculus.

EXAMPLE

Let's show the Product Rule $d(xy) = y\ dx + x\ dy$ and Quotient Rule $d(x/y) = \frac{ydx - xdy}{y^2}$ using differentials.

Leibniz' Approximation of the Sine Function
Consider the following diagram



It turns out that $dt = \frac{xdx}{\sqrt{1-x^2}}$ and by Pythagoras' Theorem $(dy)^2 = (dx)^2 + (dt)^2$

But we can combine these to show that $(dy)^2 = (dx)^2 + x^2(dy)^2$

Treating *dy* as a constant and applying the differential operator *d* to both sides produces

$$0 = d[(dx)^2 + x^2(dy)^2]$$
$$0 = 2d(dx)dx \ + 2xdx(dy)^2$$
$$0 = d(dx) + x(dy)^2$$
$$-x = \frac{d(dx)}{(dy)^2}$$

Leibniz solves the differential equation by assuming the solution is a power series of the form $x = a + by + cy^3 + dy^5 + ey^7 + \ldots$ where x(0)=0, plugging the form into both side of the equations and equating like terms produces the equations

$$\begin{aligned} 2\cdot 3c &= -b \\ 4\cdot 5d &= -c \\ 6\cdot 7e &= -d \\ 8\cdot 9f &= -e \end{aligned}$$

Which when solved leads to

$$x = \sin(y) = y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \ldots$$

GroupWork
Confirm this result.