History of Mathematics

Math 395 Spring 2010 — Fowler 310 MWF 10:30am - 11:25am

 — http://faculty.oxy.edu/ron/math/395/10/

Class 22: **Monday March 31**

TITLE Infinitesimally Close To Calculus
CURRENT READING: Katz, §15
NEXT READING: Katz, §16

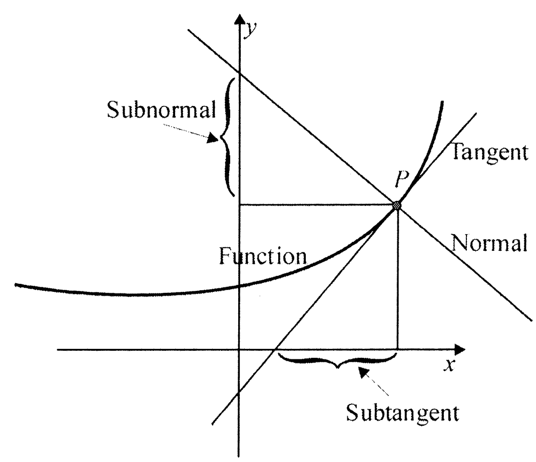
Homework #8 DUE Friday April 2 (in class)
Katz, p. 418: #4, #8, #21, #30, #37. EXTRA CREDIT: page 419, #29.

SUMMARY
We will discuss the work of people who did work which led to the development of the Calculus prior to Newton and Leibniz.

The Beginnings of Calculus

Prior to the work of Newton and Leibniz there was a great deal of work on early concepts that we now associate with calculus, some of which are:

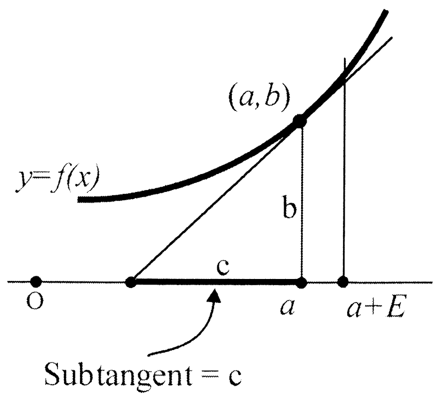
- Infinitesimal analysis
- Tangents and Normals
- Maxima and Minima
- Areas and Volumes



Fermat's *Method of Adequality*
In 1629, Pierre de Fermat (1601-1665) wrote in *Method of Finding Maxima and Minima,* an algorithm he called the method of adequality which he used to find the extrema of functions and the slopes of tangents to curves.

Extrema
1) Given *f(x),* compute *f(x+E)* and adequate the two (set them equal to each other)
2) Cancel common terms and solve for *x* after removing any term that contains E.

EXAMPLE
Use Fermat's method of adequality to find the location of the maximum of $f(x) = -x^2 + 3x - 2$



Fermat gives the formula ("the adequality") needed to find the tangent to a curve $y=f(x)$ at (a,b) as

$$\frac{b}{c} = \frac{f(a+E)}{c+E}$$

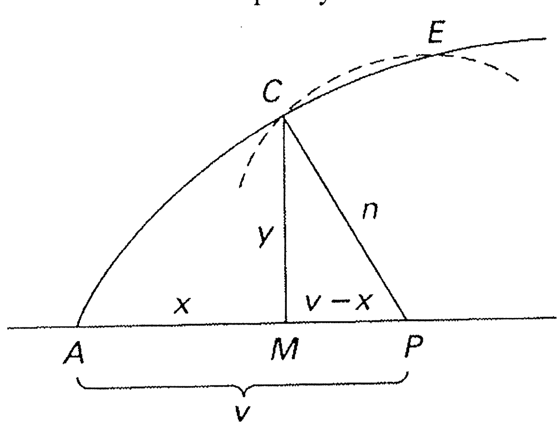
1) Cross multiply and cancel terms
2) Divide by E
3) Solve for c (ignoring terms with E or setting E=0)

Exercise

Use Fermat's method of adequality to define the tangent to $f(x) = x^2 - 2x + 3$ at $a=2$

René Descartes' Method of Normals

In *la géométrie* Descartes proposed a method to calculate normals to a curve at a specific point. Since tangents are perpendicular to normals, this was equivalent, and in Descartes mind, superior to the method of adequality of Fermat.



Descartes wants **P** to be the center of a tangent circle and thus sets up the equation

$$[f(x)]^2 + (v-x)^2 - n^2 = 0$$

Where

$$[f(x)]^2 + v^2 - 2vx + x^2 - n^2 = (x - x_0)^2 q(x) = 0$$

And solve v in terms of x_0 after balancing terms (often called the Method of Undetermined Coefficients)

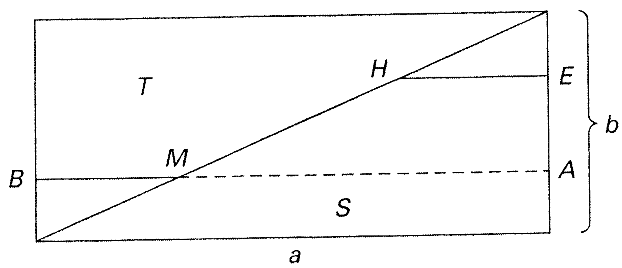
EXAMPLE

Use Decartes method of normals to obtain the slope of the normal to $y = x^2$ at the point (x_0, x_0^2) by obtaining the equations

$$\begin{aligned} a - 2x_0 &= 0 \\ b - 2x_0 a + x_0^2 &= 1 \\ a x_0^2 - 2b x_0 &= -2v \\ b x_0^2 &= v^2 - n^2 \end{aligned}$$

Integral Calculus

Bonaventura Cavalieri (1598-1647) was a student of Galileo's and the first to develop a coherent theory of working with "indivisibles" (infinitesimals). In 1635 he published *Geometry, Advanced in a New Way by the Indivisibles of the Continua* in which he introduced the concept of *omnes lineas* or "all the lines" of a plane figure F, which he denoted $\mathcal{O}_F(l)$. Cavalieri meant "the collection of intersections of the plane figure with a perpendicular plane moving parallel to itself from one side of the given figure to the other."



Each line segment *BM* in triangle T corresponds to a line segment *HE* in triangle S

$$\mathcal{O}_T(l) = \mathcal{O}_S(l)$$

Every segment BA from the rectangle is made up a line from triangle S and one from triangle T

$$\mathcal{O}_F(l) = \mathcal{O}_T(l) + \mathcal{O}_S(l)$$

Together these two equations imply that

$$\mathcal{O}_F(l) = 2\mathcal{O}_T(l)$$

Using modern calculus symbols, this result is equivalent to

$$ab = 2\int_0^b \frac{a}{b}t\ dt$$

Which becomes

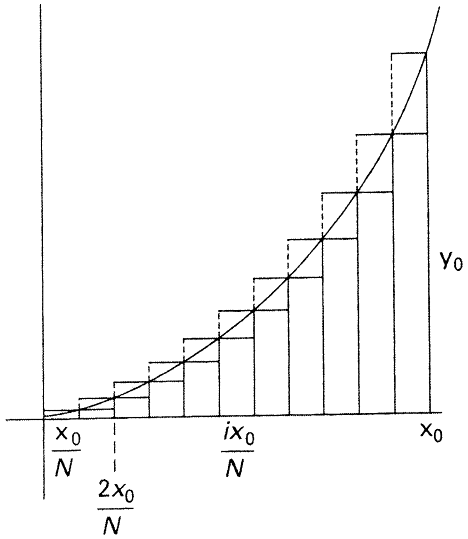
$$b^2 = 2\int_0^b t\ dt$$

By 1647 Cavalieri had been able to expand his method of indivisibles to compute the area under what was known as the "higher parabola" $y = x^k$:

$$\int_0^b x^k\ dx = \frac{1}{k+1}b^{k+1}$$

Katz claims that this result was known by many others working in the pre-Calculus mid 17th century such as Fermat, Pascal, Evangelista Torricelli (1608-1647) and Gilles Persone de Roberval (1602-1675)

The Area Under Any Higher Parabola



Divide the interval from x=0 to x=x_0 into N subintervals under the curve $y=px^k$ produces the expression for the sum of the areas of the rectangles

$$p\frac{x_0^k}{N^k}\frac{x_0}{N}+p\frac{(2x_0)^k}{N^k}\frac{x_0}{N}+\cdots+p\frac{(Nx_0)^k}{N^k}\frac{x_0}{N}=\frac{px_0^{k+1}}{N^{k+1}}\left(1^k+2^k+\cdots+N^k\right)$$

Which leads to inequalities for the exact area A

$$\frac{px_0^{k+1}}{N^{k+1}}(1^k+2^k+\cdots+(N-1)^k)<A<\frac{px_0^{k+1}}{N^{k+1}}(1^k+2^k+\cdots+N^k)$$

Considering that the difference between these two estimates goes to zero as N increases the exact area A is

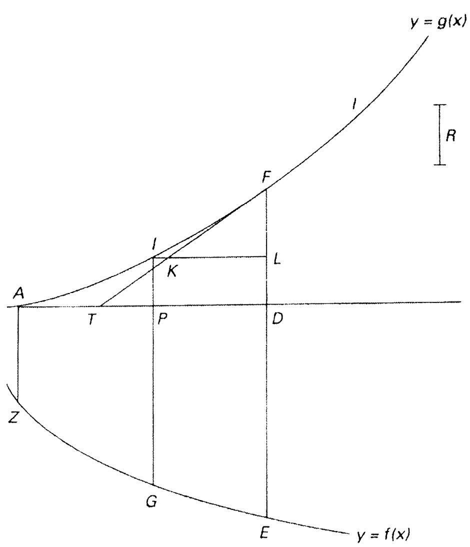
$$\frac{px_0^{k+1}}{k+1}=\frac{x_0y_0}{k+1}$$

Both Roberval and Fermat used this method to compute this area. Pascal used his knowledge of the Pascal's triangle to compute $\sum_{i=1}^{N} i^k$

and the result

$$\sum_{i=1}^{N-1} i^k<\frac{N^{k+1}}{k+1}<\sum_{i=1}^{N} i^k$$

Fermat was able to find the areas under "higher hyperbolas," i.e. curves of the form $y^m x^k = p$ in his *Treatise on Quadrature*. **John Wallis** (1616-1703), a British mathematician who taught at Oxford was able to compute the area under curves of the form $y = x^{p/q}$ where p and q are integers. **Isaac Barrow** (1630-1677) and **James Gregory** (1638-1675) understood versions of the Fundamental Theorem of calculus. Barrow was Lucasian professor of Mathematics at Cambridge while Newton attended there.



Barrow defines

$$Rg(x)=\int_a^x f(x)dx$$

$$g'(x)=\frac{g(x)}{t(x)}=\frac{f(x)}{R} \text{ or } \frac{d}{dx}\int_a^x f(x)\,dx=f(x)$$

$$\int_a^b Rf'(x)dx=R(f(b)-Rf(a))$$